CS 198-087 Introduction to Mathematical Thinking
Fall 2018

Final Solutions

This is the final examination for Introduction to Mathematical Thinking.

- This exam has 8 questions (including question 0). The exam is out of 64 points.
- The exam will last for exactly 1 hour and 20 minutes, unless you have pre-arranged DSP accomodations.
- Fit all of your answers in the space provided.
- You are allowed to consult two double-sided, hand-written cheat sheets, but nothing else. No electronics.

DO NOT TURN THE PAGE UNTIL INSTRUCTED.

In the meantime, fill out the information on this page.

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0 Preliminary Questions

Points: 2 (1 each)

a) On a scale of 1 to 10, how are you feeling about this exam?

10, hopefully

b) What was your favorite topic covered in this course?

All of it!

1 ZOBOOMAFOO

Points: 14 (2/4/4/4)

a) Determine the number of permutations of ZOBOOMAFOO.

In ZOBOOMAFOO, there is 1 A, 1 B, 1 F, 1 M, 1 Z, and 5 Os. The number of permutations of ZOBOOMAFOO is then the number of total characters factorial, divided by the number of instances factorial, for each repeated character. Numerically, this is

$$\boxed{\frac{10!}{5!}}$$

b) Determine the number of permutations of ZOBOOMAFOO, where "ZBMA" appear next to each other, in any order. (e.g. "ZAMB", "BMAZ" appear as substrings)

Now, we treat "ZBMA" as a single character. This means we have 7 characters — 1 "ZBMA", 1 F, and 5 Os. If all we wanted to count was the number of permutations where ZBMA appear exactly in that order, our result would simply be $\frac{7!}{5!}$, however we need to account for all of the possible orderings of ZBMA.

There are 4! permutations of ZBMA. Therefore, our final result is

$$\boxed{\frac{7!4!}{5!}}$$

c) Determine the number of permutations of ZOBOOMAFOO, where the letters Z, B, M, A, F appear in alphabetical order. *(Hint: How can you model this using stars and bars?)*

We can think of this as being a stars and bars problem, where the 5 Os are our stars, and A, B, F, M, Z are our bars. This is because the order of A, B, F, M, Z is fixed; all that varies is the number of Os in between them.

Here, there are 5 stars, and 5 bars, yielding a result of

$$\binom{5+5}{5} = \boxed{\binom{10}{5}}$$

d) Determine the number of three-letter strings made up of characters from ZOBOOMAFOO. (e.g. "ZOO", "MBF", "OOO", "ZOF")

There are three cases here:

- Case 1: 3 Os
- Case 2: 2 Os and 1 other character
- Case 3: All 3 characters are distinct

Case 1: All 3 Os

There is only **1** such three-letter string, namely, "OOO".

Case 2: 2 Os, 1 other character

Here, there are three possibilities, "OOX", "OXO", and "XOO", where X can be replaced by one of the other 5 characters (A, B, F, M, Z).

There are 3 subcases, each of which have 5 options (one of the five letters), meaning the total number of three-letter strings in Case 2 is $3 \cdot 5 = \mathbf{15}$.

Case 3: 3 unique characters

If all three characters are unique, there are 6 options for the first character (A, B, F, M, Z, O), 5 options for the second character and 4 options for the third character.

Therefore, in Case 3, there are $6 \cdot 5 \cdot 4 = \mathbf{120}$ such three-letter strings.

Our total number of three-letter strings is then $1 + 15 + 120 = \boxed{136}$.

2 Combinatorial Proofs

Points: 8

(This problem was modified after the start of the exam, this document reflects the updated version of these problems.)

Give a combinatorial proof of the following statement:

$$\binom{n}{k}\binom{k}{j} = \binom{n}{j}\binom{n-j}{k-j}$$

Suppose we want to select a team of k basketball players from a pool of n, and suppose we want our team to have j "captains".

LHS: First, we can select our k team members, which can be done in $\binom{n}{k}$ ways. Then, from the k team members we need to select j captains, which can be done in $\binom{k}{j}$ ways. Our result is then the product $\binom{n}{k}\binom{k}{j}$.

RHS: First, we select our j captains, which can be done in $\binom{n}{j}$ ways. Then, from the remaining $n-j$ people, we still need to select $k-j$ non-captain players, which can be done in $\binom{n-j}{k-j}$ ways. Our result is then $\binom{n}{j}\binom{n-j}{k-j}$.

Since both descriptions count the same quanitity, they must be equal. QED.

3 Primality

Points: 8

Prove that if p is a prime, $p \geq 5$, then $p = 6k+1$ or $p = 6k-1$ for some $k \in \mathbb{N}$.

We present two solutions.

Solution 1: Direct

We know that p is some prime greater than or equal to prime. We know p must be odd, as the only even prime is two. That means both $p+1$ and $p-1$ must be even. Additionally, since we know that in any three consecutive integers, exactly one must be a multiple of 3, we know that either $p+1$ is a multiple of 3, or $p-1$ is a multiple of 3 (it cannot be p as p is prime).

If $p+1$ is a multiple of 3, since we've already established that $p+1$ is also even, we have that $p+1$ is a multiple of 6, i.e. $p+1 = 6k$, $k \in \mathbb{Z}$. However, this implies that $p = 6k-1$.

If $p-1$ is a multiple of 3, since we've already established that $p-1$ is also even, we have that $p-1$ is a multiple of 6, i.e. $p-1 = 6k$, $k \in \mathbb{Z}$. However, this implies that $p = 6k+1$.

Therefore, either $p = 6k+1$ or $p = 6k-1$ for some positive integer k.

Solution 2: Contrapositive

The contrapositive of this statement is the statement, "If $p \neq 6k+1$ and $p \neq 6k-1$, then p is not prime."

If $p \neq 6k+1$ and $p \neq 6k-1$, then we know that p is not equivalent to either 1 or 5 in modulo 6. This means that p is either equivalent to 0, 2, 3, or 4, in modulo 6. This means that either $p = 6k$, $p = 6k+2$, $p = 6k+3$ or $p = 6k+4$.

- If $p = 6k$, clearly p is not prime, as it is a multiple of 6.
- If $p = 6k+2$, we can rewrite it as $p = 2(3k+1)$, which tells us it is a multiple of 2, and hence not prime
- If $p = 6k+3$, we can rewrite it as $p = 3(2k+1)$, which tells us it is a multiple of 3, and hence not prime
- If $p = 6k+4$, we can rewrite it as $p = 2(3k+2)$, which tells us it is a multiple of 2, and hence not prime

Therefore, if $p \neq 6k+1$ and $p \neq 6k-1$, p cannot be prime, and by contraposition, the original statement holds.

4 Modular Arithmetic, Mechanical

Points: 8 (4/4)

a) Evaluate $15^{26} \pmod{23}$.

By Fermat's Little Theorem, we know that $a^{22} \equiv 1 \pmod{23}$, since 23 is prime. Therefore, $15^{22} \equiv 1 \pmod{23}$. Then, we can write 15^{26} as $15^{22} \cdot 15^4$.

$$\begin{aligned} 15^{22} \cdot 15^4 &\equiv 1 \cdot 15^4 \\ &\equiv 225^2 \\ &\equiv (-5)^2 \\ &\equiv 25 \\ &\equiv \boxed{2} \pmod{23} \end{aligned}$$

b) Determine $17^{-1} \pmod{63}$.

Recall, our goal is to find x such that $17x + 63y = 1$.

Our calls to the extended Euclidean algorithm look like gcd(63, 17) = gcd(17, 12) = gcd(12, 5) = gcd(5, 2) = gcd(2, 1).

Writing out our relationships from the division algorithm yields

$$\begin{aligned} 63 &= 3 \cdot 17 + 12 \\ 17 &= 1 \cdot 12 + 5 \\ 12 &= 2 \cdot 5 + 2 \\ 5 &= 2 \cdot 2 + 1 \end{aligned}$$

Rearranging for the remainders yields

$$\begin{aligned} 12 &= 63 - 3 \cdot 17 \\ 5 &= 17 - 1 \cdot 12 \\ 2 &= 12 - 2 \cdot 5 \\ 1 &= 5 - 2 \cdot 2 \end{aligned}$$

Substituting into the last equation:

$$\begin{aligned} 1 &= 5 - 2 \cdot 2 \\ &= 5 - 2 \cdot (12 - 2 \cdot 5) = 5 \cdot 5 - 2 \cdot 12 \\ &= 5 \cdot (17 - 1 \cdot 12) - 2 \cdot 12 = 5 \cdot 17 - 7 \cdot 12 \\ &= 5 \cdot 17 - 7 \cdot (63 - 3 \cdot 17) = 26 \cdot 17 - 7 \cdot 63 \end{aligned}$$

Therefore, $\boxed{26}$ is the inverse of 17 in modulo 63.

5 Fun...ctions

Points: 8 (3/5)

Suppose $f_k(x) = (x-1)(x-2)....(x-k)$, where k is some odd integer.

a) What is the coefficient on x^{k-1}? (Your answer should be a function of k.)

The coefficient on the second-highest power of x is always the negative of the sum of the roots. In this case, the roots are $1, 2, 3, ..., k$. Therefore, the coefficient on x^{k-1} is $-(1+2+3+...+k) = \boxed{-\frac{k(k+1)}{2}}$.

b) What is the coefficient on x? (Your answer should be a function of k. You can leave it as a sum.)

The coefficient on x is always the sum of the product of the roots of f, taken $k-1$ at a time. Recall, the degree 3 case:

$$f(x) = (x-1)(x-2)(x-3) = x^3 - (1+2+3)x^2 + (1\cdot 2 + 1\cdot 3 + 2\cdot 3)x - 1\cdot 2\cdot 3$$

Here, $k = 3$, and the coefficient on x is the sum of the product of the roots, taken 2 at a time. Notice, we can also write this sum as $\frac{3!}{1} + \frac{3!}{2} + \frac{3!}{3}$.

In the general k case, this sum will look like

$$\frac{k!}{1} + \frac{k!}{2} + \frac{k!}{3} + ... + \frac{k!}{k}$$

which we can simplify to be

$$\boxed{k!\left(\sum_{i=1}^{k} \frac{1}{i}\right)}$$

6 Poly No Meal

Points: 8 (2/4/2)

Let $f(x) = (x^5 - 2x^{-3})^{12}$. Determine each of the following.

a) The sum of the coefficients in the expansion of $f(x)$

The sum of the coefficients is given by $f(1)$, which in this case is $(1^5 - 2\cdot 1^{-3})^{12} = (1-2)^{12} = (-1)^{12} = \boxed{1}$.

b) The general term t_k in the expansion of $f(x)$

$$\begin{aligned} t_k &= \binom{12}{k}(x^5)^{12-k}(-2x^{-3})^k \\ &= (-1)^k\binom{12}{k}2^k x^{60-5k}\cdot x^{-3k} \\ &= \boxed{(-1)^k\binom{12}{k}2^k x^{60-8k}} \end{aligned}$$

c) The coefficient on x^{20} in the expansion of $f(x)$

First, we set the exponent on x from the previous part to 20, and solve for k.

$$60 - 8k = 20 \implies k = 5$$

Then, we substitute $k = 5$ into the general term:

$$t_5 = (-1)^5\binom{12}{5}2^5 x^{20}$$

which tells us the coefficient on x^{20} is $\boxed{-32\binom{12}{5}}$.

7 Polynomial Interpolation

Points: 8 (4/2/2)

Suppose we want to find the polynomial that interpolates $\{(1,5),(2,6),(4,1)\}$ using Lagrange Interpolation.

a) Find $p_1(x)$, the sub-polynomial corresponding to $x_1 = 1$.

$$p_1(x) = \frac{(x-2)(x-4)}{(1-2)(1-4)} = \boxed{\frac{x^2-6x+8}{3}}$$

b) Now, suppose we want to find the interpolating polynomial under mod q, for some q. Why cannot we do this when $q = 12$? Give a concrete example of a calculation that cannot be done in mod 12.

Not all denominators have inverses in mod 12. For example, in $p_1(x)$ above, there is no inverse of 3. Therefore, we cannot find $p_1(x)$, and cannot find $p(x)$.

c) For some q, the interpolating polynomial is $p(x) \equiv x+4 \pmod{q}$. Determine q. Justify your answer.

$\boxed{q=7}$. We can initially make this guess by noticing a pattern $(1,5),(2,6),(3,0),(4,1)$ is linear with a slope of 1 and offset of 4.

To verify:

$$p(1) = 1+4 \equiv 5 \pmod{7}$$
$$p(2) = 2+4 \equiv 6 \pmod{7}$$
$$p(4) = 4+4 = 8 \equiv 1 \pmod{7}$$