CS 198-087 Introduction to Mathematical Thinking
Fall 2018

Midterm

This is the midterm examination for Introduction to Mathematical Thinking.

- This exam has 7 questions (including question 0). The exam is out of 72 points.
- The exam will last for exactly 1 hour and 20 minutes, unless you have pre-arranged DSP accomodations.
- You can use the backs of pages for scrap work, but please write your answers only on the fronts of pages.

DO NOT TURN THE PAGE UNTIL INSTRUCTED.

In the meantime, fill out the information on this page.

Name:

@berkeley.edu email:

Student ID Number:

Name of student to your left:

Name of student to your right:

0 Preliminary Questions

Points: 3 (1 each)

a) On a scale of 1 to 10, how are you feeling about this exam?

b) What is your favorite topic so far in this course?

c) Name one of the songs I played in class on Monday.

1 True or False

Points: 17 (1 each)

Circle either true or false in each of the below.

a) **True** or **False**: $\forall n \in \mathbb{N}, n^2 + 1 < 0 \implies n^2 + 2 < 0$

b) **True** or **False**: $\sum_{i=1}^{n} i = \frac{n^2+n}{2}$

c) **True** or **False**: $P \oplus Q \equiv (P \vee Q) \wedge (\neg P \wedge \neg Q)$

d) **True** or **False**: The contrapositive of the statement "if x is even, then x^2 is odd" is "if x^2 is even, then x is odd."

e) **True** or **False**: $P \iff Q \equiv (P \implies Q) \wedge (Q \implies P)$

f) **True** or **False**: $\forall n \in \mathbb{Z}^+, 3 | n^3 + 2n$ *(Hint: Try and prove or disprove the statement.)*

g) **True** or **False**: The set of all even integers is countably infinite.

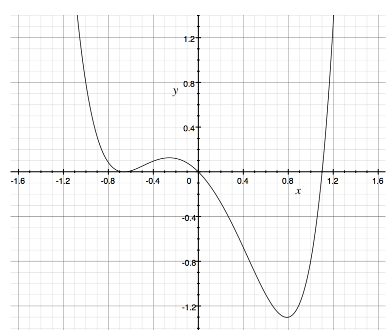
h) **True** or **False**: The union of ten countably infinite sets is uncountably infinite.

i) **True** or **False**: A function is surjective if $\forall a \in A, \exists b \in B : f(a) = b$.

j) **True** or **False**: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f : x \mapsto 2x^3 - 15$ is surjective.

k) **True** or **False**: If A, B are two disjoint sets, then $|A \cup B| = |A - B| + |B - A|$.

l) **True** or **False**: The following function is injective.



For the remaining parts, consider sets $A = \{1,2,3,4,5\}$ and $B = \{1,2,3\}$.

m) **True** or **False**: There exists a bijection $f : A \to B$.

n) **True** or **False**: The relation $r : \{(1,3),(3,2),(2,2),(4,1),(5,1)\}$ is a function.

o) **True** or **False**: $B \subseteq A$

p) **True** or **False**: $(B \cup \{6\}) \subset A$

q) **True** or **False**: $|A - B| = |A| - |B|$

2 Set Matching

Points: 10 (2 each)

Consider the following sets, where the universe is $\mathbb{N}$:

$A = \{x : x \text{ is prime}\}$

$B = \{x : x = a^3, a \in \mathbb{N}\}$

$C = \{2, 15, 18, 64\}$

$D = \{15, 49, 81\}$

Using the above sets and any set operations, construct the following sets. For example, to create the set $\{15\}$, we can do the operation $C \cap D$. (There may be more than one potential answer, but you only need to identify one.)

a) $\emptyset$ (the empty set)

b) $\{15, 18, 49, 81\}$

c) $\{2\}$

d) $\{1, 2\}$

e) $\{x : \exists y \in \mathbb{N} : (1 < y < x) \wedge y | x\}$

3 Set Proof

Points: 6

Prove that if A and B are sets, then $A \cap (B - A) = \emptyset$.

4 Choose One

Points: 8 (+2 for completing both)

Do either part a) or part b) below. If you complete both, you will earn some extra credit, but only worry about that if you've completed the rest of the exam.

a) Prove that $\sqrt{3}$ is irrational.

b) Prove that there are no integer solutions to $x^2 - 3 = 4y$. *(Hint: Break x into four cases.)*

5 Induction

Points: 16 (8 each)

a) Prove that $8|9^n - 1$, for all $n \in \mathbb{N}$.

b) Prove that $\sum_{i=0}^{n} 2^{-i} = 2 - 2^{-n}$.

6 Fun with Logic

Points: 12 (a: 2, b: 4, c: 6)

In this question we will explore the NOR logical operator, sometimes represented with the symbol $\downarrow$. The operation $A \downarrow B$ is true only when both A, B are false. It turns out that $A \downarrow B$ is complete, i.e. we can rewrite any other logical operation using only $\downarrow$. *(Fun fact: The computer that was used on the Apollo mission to first send humans to the moon was programmed using only NOR gates!)*

a) Rewrite $A \downarrow B$ in terms of $\neg, \vee, \wedge$, and use a truth table to prove your result. *(Hint: You may not need to use all three).*

b) Rewrite the standard negation operation $\neg A$ using only $\downarrow$, and use a truth table to prove your result.

c) Rewrite the standard disjunction operation $A \vee B$ using only $\downarrow$, and use a truth table to prove your result. *(Hint: We can rewrite the standard conjunction $A \wedge B$ as $(A \downarrow A) \downarrow (B \downarrow B)$).*