CS 198-087 Fall 2018 | Introduction to Mathematical Thinking | Midterm Solutions

This is the midterm examination for Introduction to Mathematical Thinking.

- This exam has 7 questions (including question 0). There are 55 total points.
- The exam will last for exactly 1 hour and 20 minutes, unless you have pre-arranged DSP accomodations.
- You can use the backs of pages for scrap work, but please write your answers only on the fronts of pages.

DO NOT TURN THE PAGE UNTIL INSTRUCTED.

In the meantime, fill out the information on this page.

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0 Preliminary Questions

a) On a scale of 1 to 10, how are you feeling about this exam? Hopefully, a 10!

b) What is your favorite topic so far in this course? Everything!

c) Name one of the songs I played in class on Monday. Chammak Challo.

1 True or False

Circle either true or false in each of the below.

a) **True** or **False**: $\forall n \in \mathbb{N}, n^2+1<0 \implies n^2+2<0$

True. $n^2+1<0$ is never true for any natural number, and any implication where the first clause (P) is false is true.

b) **True** or **False**: $\sum_{i=1}^{n} i = \frac{n^2+n}{2}$

True. We discussed this in lecture, and it was on the homework.

c) **True** or **False**: $P \oplus Q \equiv (P \vee Q) \wedge (\neg P \wedge \neg Q)$

False. $P \oplus Q \equiv (P \vee Q) \wedge \neg(P \wedge Q)$ which simplifies to $(P \vee Q) \wedge (\neg P \vee \neg Q), which is not what the question states.$

d) **True** or **False**: The contrapositive of the statement "if x is even, then x^2 is odd" is "if x^2 is even, then x is odd."

True. The negation of "x^2 is odd" is "x^2 is even", and the negation of "x is even" is "x is odd."

e) **True** or **False**: $P \iff Q \equiv (P \implies Q) \wedge (Q \implies P)$

True. The statement "P if and only if Q" is true only when both $P \implies Q$ and $Q \implies P$.

f) **True** or **False**: $\forall n \in \mathbb{Z}^+, 3 | n^3+2n$ *(Hint: Try and prove or disprove the statement.)*

True. You can prove this by considering three cases for x: $x=3k$, $x=3k+1$ and $x=3k+2$.

g) **True** or **False**: The set of all even integers is countably infinite.

True. The set of all integers is countable, and so a subset of a countable set is also countable.

h) **True** or **False**: The union of ten countably infinite sets is uncountably infinite.

False. We took up a similar question in lecture; the union of any number of countable sets is countable.

i) **True** or **False**: A function is surjective if $\forall a \in A, \exists b \in B : f(a) = b$.

False. The definition of surjective is $\forall b \in B, \exists a \in A : f(a) = b$.

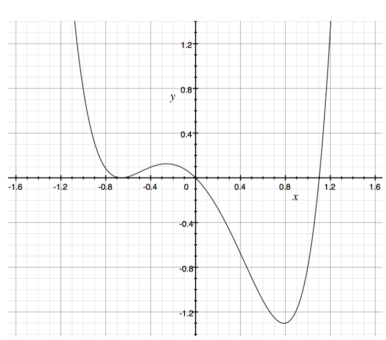
j) **True** or **False**: The function $f : \mathbb{R} \to \mathbb{R}$ given by $f : x \mapsto 2x^3 - 15$ is surjective.

True. Consider the graph of $x \mapsto x^3$ (which this is just a stretching and translation of). $f(x) \to \infty$ as $x \to \infty$, and $f(x) \to -\infty$ as $x \to -\infty$. All values in $\mathbb{R}$ are seen as outputs.

k) **True** or **False**: If A, B are two disjoint sets, then $|A \cup B| = |A - B| + |B - A|$.

True. Draw a picture to see why this is true.

l) **True** or **False**: The following function is injective.



False. It fails the horizontal line test. For example, $f(x) = 0$ has multiple solutions.

For the remaining parts, consider sets $A = \{1,2,3,4,5\}$ and $B = \{1,2,3\}$.

m) **True** or **False**: There exists a bijection $f : A \to B$.

False. It's only possible for there to be a bijection between two sets when they have the same cardinality, but $|A| = 5 \neq |B| = 3$.

n) **True** or **False**: The relation $r : \{(1,3),(3,2),(2,2),(4,1),(5,1)\}$ is a function.

True. There is only one ordered pair for each element in A.

o) **True** or **False**: $B \subseteq A$

True.

p) **True** or **False**: $(B \cup \{6\}) \subset A$

False. The element 6 is not in A.

q) **True** or **False**: $|A - B| = |A| - |B|$

True. $|A-B| = |\{4,5\}| = 2$, and $|A| - |B| = 5 - 3 = 2$. Note that this is not true in general, but it happens to be true in this case because $B \subset A$.

2 Set Matching

Consider the following sets, where the universe is $\mathbb{N}$:

$A = \{x : x \text{ is prime}\}$

$B = \{x : x = a^3, a \in \mathbb{N}\}$

$C = \{2, 15, 18, 64\}$

$D = \{15, 49, 81\}$

Using the above sets and any set operations, construct the following sets. For example, to create the set $\{15\}$, we can do the operation $C \cap D$. (There may be more than one potential answer, but you only need to identify one.)

a) $\emptyset$ (the empty set)

$A \cap B$, for example.

b) $\{15, 18, 49, 81\}$

$((C - B) - A) \cup D$. To isolate $\{15, 18\}$, we take $(C - B) - A$. We can union this with D to get the desired set.

c) $\{2\}$

$C - A^C$. Everything in C is composite other than 2.

d) $\{1, 2\}$

e) $\{x : \exists y \in \mathbb{N} : (1 < y < x) \wedge y | x\}$

A^C. This is the set of all composite numbers.

3 Set Proof

Prove that if A and B are sets, then $A \cap (B - A) = \emptyset$.

Proof by Contradiction.

Suppose there exists some x such that $x \in A \cap (B - A)$. Then, $x \in A$ and $x \in (B - A)$. But, if $x \in B - A$, then it must be the case that $x \in B$ and $x \notin A$. However, we just assumed that $x \in A$; this is a contradiction.

Therefore, by contradiction, $A \cap (B - A) = \emptyset$.

4 Choose One

Do either part a) or part b) below. If you complete both, you will earn some extra credit, but only worry about that if you've completed the rest of the exam.

a) Prove that $\sqrt{3}$ is irrational.

Proof by Contradiction.

Assume $\sqrt{3} = \frac{a}{b}$, for $a, b \in \mathbb{Z}$, where $\frac{a}{b}$ is the reduced fraction (i.e. a, b share no common factors.

Squaring both sides, we have $\frac{a^2}{b^2} = 3$, i.e. $a^2 = 3b^2$. This tell us that a^2 must be a multiple of 3. However, this also must mean that a is also a multiple of 3 (this can be proven by casework). This means we can write $a = 3k$, for $k \in \mathbb{Z}$.

Then, we have $(3k)^2 = 3b^2 \implies 9k^2 = 3b^2 \implies b^2 = 3k^2$, telling us that b^2 is a multiple of 3, meaning that b is a multiple of 3.

Now we have that both a and b are multiples of 3. This means that the fraction $\frac{a}{b}$ is not in reduced form, since we can further divide the numerator and denominator by 3. This contradicts our original assumption that a, b shared no factors.

Therefore, by contradiction, we have that $\sqrt{3}$ is irrational.

b) Prove that there are no integer solutions to $x^2 - 3 = 4y$. *(Hint: Break x into four cases.)*

Let's consider 4 cases: $x = 4k$, $x = 4k+1$, $x = 4k+2$ and $x = 4k+3$, for $k \in \mathbb{Z}$. We will also assume that y is an integer.

Case 1: $x = 4k$

$$\begin{aligned}(4k)^2 - 3 &= 4y \\ 16k^2 - 4y &= 3 \\ 2(8k^2 - 2y) &= 3\end{aligned}$$

This is contradictory, as on the left hand side we have an even number $(2 \cdot (\text{some integer}))$, and on the right hand side we have 3, an odd number. This is not possible.

Case 2: $x = 4k+1$

$$\begin{aligned}(4k+1)^2 - 3 &= 4y \\ 16k^2 + 8k + 1 - 3 &= 4y \\ 2(8k^2 + 4k - 1) &= 2(2y) \\ 8k^2 + 4k - 1 &= 2y \\ 2(4k^2 + 2k) - 1 &= 2y\end{aligned}$$

Again, this is contradictory, as on the right hand side we have an even number $(2 \cdot (\text{some integer}))$, and on the left hand side we have $2 \cdot (\text{some integer}) - 1$, which is an odd integer. This is not possible.

Case 3: $x = 4k+2$

$$\begin{aligned}(4k+2)^2 - 3 &= 4y \\ 16k^2 + 16k + 4 - 3 &= 4y \\ 2(8k^2 + 8k) + 1 &= 4y\end{aligned}$$

Again, this is contradictory, as on the left hand side we have an odd number and on the right hand side we have an even number.

Case 4: $x = 4k+3$

$$\begin{aligned}(4k+3)^2 - 3 &= 4y \\ 16k^2 + 24k + 9 - 3 &= 4y \\ 2(8k^2 + 12k + 3) &= 2(2y) \\ 8k^2 + 12k + 2 + 1 &= 2y \\ 2(4k^2 + 6k + 1) + 1 &= 2y\end{aligned}$$

Again, this is contradictory, as on the left hand side we have an odd number and on the right hand side we have an even number.

In all four cases we've shown a contradiction, meaning that it is not possible for there to be solutions for x and y in the integers.

5 Induction

a) Prove that $8|9^n - 1$, for all $n \in \mathbb{N}$.

Base Case: $n = 1$ $8|9^1 - 1$, i.e. $8|8$.

Since $8|8$, the base case holds.

Induction Hypothesis: $n = k$

Assume that $8|9^k - 1$ for some arbitrary $k \in \mathbb{Z}$. We can also write this information as $9^k - 1 = 8c$, for some constant $c \in \mathbb{Z}$.

Induction Step

Now, we need to show that $8|9^{k+1} - 1$, i.e. that there's some constant $b \in \mathbb{Z}$ such that $9^{k+1} - 1 = 8b$.

$$
\begin{aligned}
9^{k+1} - 1 &= 9 \cdot 9^k - 1 \\
&= 9 \cdot 9^k - 9 + 9 - 1 \\
&= 9(9^k - 1) + 8 \\
&= 9(8c) + 8 \\
&= 8(9c + 1)
\end{aligned}
$$

i.e., $8|9^{k+1} - 1$.

We've now shown that if $8|9^k - 1$, then $8|9^{k+1} - 1$; therefore, by induction, the statement holds.

b) Prove that $\sum_{i=0}^{n} 2^{-i} = 2 - 2^{-n}$.

Base Case: $n = 0$

LS: $\sum_{i=0}^{0} 2^{-i} = 2^{-0} = 1$ RS: $2 - 2^{-0} = 2 - 1 = 1$

Since LS = RS, we have that the base case holds.

Induction Hypothesis: $n = k$ Assume that $\sum_{i=0}^{k} 2^{-i} = 2 - 2^{-k}$, for some arbitrary $k \in \mathbb{N}$.

Induction Step We now want to show that $\sum_{i=0}^{k+1} 2^{-i} = 2 - 2^{-(k+1)}$. Let's start with the left side and work towards the right side.

$$
\begin{aligned}
\sum_{i=0}^{k+1} 2^{-i} &= \sum_{i=0}^{k} 2^{-i} + 2^{-(k+1)} \\
&= 2 - 2^{-k} + 2^{-(k+1)} \\
&= 2 - \frac{2}{2 \cdot 2^k} + \frac{1}{2^{k+1}} \\
&= 2 - \frac{1}{2^{k+1}}
\end{aligned}
$$

as required. Therefore, by induction, the statement holds.

6 Fun with Logic

In this question we will explore the NOR logical operator, sometimes represented with the symbol $\downarrow$. The operation $A \downarrow B$ is true only when both A, B are false. It turns out that $A \downarrow B$ is complete, i.e. we can rewrite any other logical operation using only $\downarrow$. *(Fun fact: The computer that was used on the Apollo mission to first send humans to the moon was programmed using only NOR gates!)*

a) Rewrite $A \downarrow B$ in terms of $\neg, \vee, \wedge$, and use a truth table to prove your result. *(Hint: You may not need to use all three).*

We know $A \downarrow B$ should be true only when A, B are both false. This is exactly the opposite of $A \vee B$, which is only false when both A, B are false. Therefore, we propose

$$A \downarrow B \equiv \neg(A \vee B)$$

To prove it, we use a truth table.

A	B	A $\downarrow B$	$\neg(A \vee B)$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

Therefore, we have that $A \downarrow B \equiv \neg(A \vee B)$.

b) Rewrite the standard negation operation $\neg A$ using only $\downarrow$, and use a truth table to prove your result.

We want an expression that is false when A is true, and true when A is false. We know $A \downarrow A$ is only true when both A and A (so, just A) are false, and in any other case $A \downarrow A$ is false. Hence, we propose

$$\neg A \equiv A \downarrow A$$

and once again we resort to a truth table for proof.

A	$\neg A$	A $\downarrow A$
T	F	F
F	T	T

Therefore, we have that $\neg A \equiv A \downarrow A$.

c) Rewrite the standard disjunction operation $A \vee B$ using only $\downarrow$, and use a truth table to prove your result. *(Hint: We can rewrite the standard conjunction $A \wedge B$ as $(A \downarrow A) \downarrow (B \downarrow B)$).*

We know that $A \downarrow B \equiv \neg(A \vee B)$, from part (a). By De Morgan's Laws, this means that $A \vee B \equiv \neg(A \downarrow B)$. We can now use the expression for the negation from part (b) to see that

$$A \vee B \equiv (A \downarrow B) \downarrow (A \downarrow B)$$

.

A	B	A $\vee B$	(A $\downarrow B) \downarrow (A \downarrow B)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F