PROBLEM SET 1: SET THEORY, FUNCTIONS

CS 198-087: INTRODUCTION TO MATHEMATICAL THINKING
UC BERKELEY EECS
FALL 2018

This homework is due on Wednesday, September 12th, at 6:30PM, on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. Use LaTeX if possible.

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1. Fill out the following student information form: https://goo.gl/forms/Kv5og6iAcKrO7jBj1.
2. Revisit the diagnostic for this course at http://imt-decal.org/assets/diagnostic.pdf.
   a. Which of the problems are you able to comfortably answer?
   b. Which problem did you find to be the easiest? The most difficult?
3. Let $A_1, A_2, ...A_n$ be disjoint sets such that make up the universe. That is, $\cup_{i=1}^{n} A_i = A_1 \cup A_2 \cup ... \cup A_n = \mathbb{U}$. For any other set $B \subset \mathbb{U}$, show that
$$|B| = |B \cap A_1| + |B \cap A_2| + ... + |B \cap A_n|$$
   This identity is used in deriving the total probability rule in probability theory. *(Hint: Draw a picture.)*
4. In this question, we will introduce the Principle of Inclusion-Exclusion, which allows us to measure the size of the union of two sets. We will study this more when we learn counting, as there are significant implications of PIE in combinatorics.
   a. The Principle of Inclusion-Exclusion for two sets states that $|A \cup B| = |A| + |B| - |A \cap B|$. Derive this identity. *(Hint: Draw a picture.)*
   b. The Principle of Inclusion-Exclusion for three sets states that $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. Derive this identity.
   c. (Optional) Generalize the Principle of Inclusion-Exclusion for any number of n sets. *(Hint: It may help to first derive the expression for four sets. Do you notice a pattern?)*
5. Let A, B and C be sets.
   a. Determine $|A - B| + |B - A|$ (that is, the size of the set of elements that are either in A, or in B, but not both) in terms of $|A|, |B|$ and $|A \cap B|$.
   b. Determine $|(A - B) - C| + |(B - A) - C| + |(C - A) - B|$, the size of the set of elements that are in exactly one of A, B, C in terms of the relevant quantities. *Hint: If A, B and C are disjoint, what is this quantity? Make sure your expression satisfies this case as well.*

6. Show that De Morgan's Law for sets holds; that is, verify the following:

$$(A \cap B)^C = A^C \cup B^C$$

(Hint: Consider some universe, and two sets A, B that overlap. Assign names to different subsets, and show that both sides of the equals sign count the same objects.)

7. In lecture, we showed that the composition of two injective functions is also injective, as follows:

Assume f, g are both one-to-one functions. Consider $f(g(x_1)) = f(g(x_2))$. Since $f(\cdot)$ is injective, we have that $g(x_1) = g(x_2)$. Since $g(\cdot)$ is injective, we have that $x_1 = x_2$. Therefore, we have that $f(g(x_1)) = f(g(x_2))$ implies that $x_1 = x_2$, meaning that the function $f(g(x))$ is injective.

Use a similar argument to show that the composition of two surjective functions is also surjective.

8. Use set-builder notation to describe each of the following sets. *(Hint: You may find the following definition handy: $\mathbb{N}_0 = \{0, 1, 2, 3, 4, ...\}$. Hint 2: You can specify multiple conditions when using set-builder notation.)*

   a. $\{0, 2, 4, 6, 8, 10, 12, ...\}$

   b. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ... \frac{1}{13}\}$

   c. $\{0, 1, 00, 01, 100, 101, 110, 111, 1000, ...\}$

9. Sets A, B, C are defined over a universe $\mathbb{U} = \{z : z \in \mathbb{N}_0, z \leq 25\}$ as follows:

   - $A = \{x : x \text{ is prime}, x \leq 25\}$
   - $B = \{2k : k \in \mathbb{N}_0, k \leq 25\}$
   - $C = \{t^2 : t \in \mathbb{N}_0, t \leq 25\}$

   Determine the sets that result after each of these set operations. *(Hint: A set with one element is still a set.)*

   a. $A \cap B$

   b. $(A \cup B) \cap C$

   c. $B - C$

   d. $A \setminus B^C$

   e. $A^C \cap B^C \cap C^C$

10. As we will see in Section 1.3, we have the following definitions:

   - $\mathbb{N} = \{1, 2, 3, 4, ...\}$
   - $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$
   - $\mathbb{R}_{\geq 0} =$ the set of all non-negative real numbers

Determine whether each of the following functions is injective, surjective, both (bijective) or none.

a. $f : \{2, 3, 4\} \to \{2, 3, 4\}, \{(2, 2), (3, 2), (4, 4)\}$

b. $f : \{2, 3, 4\} \to \{2, 3, 4\}, \{(2, 3), (3, 2), (4, 4)\}$

c. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$

d. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 + 3x^2 - 7x - 2$ *(Hint: Think about what the graph of f looks like.)*

e. $f : \mathbb{R}_{\geq 0} \to \mathbb{N}, f(x) = \lceil x \rceil$ *(Hint: This is the ceiling function.)*

f. $f : \mathbb{R}_{\geq 0} \to \mathbb{N}, f(x) = \lfloor x \rfloor$ *(Hint: This is the floor function.)*

g. $f : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}, f(x, y) = x^2 + y^2$

h. $f : \mathbb{N} \to \{t : t \in \mathbb{N}, t \text{ is prime}\}, f(x) = \text{the } x^{th} \text{ prime number}$