Problem Set 1: Sets and Functions

CS 198-087: Introduction to Mathematical Thinking
UC Berkeley EECS
Spring 2019

This homework is due on Friday, February 8th at 11:59PM on Gradescope. As usual, this homework is graded on effort, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX.

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1. Fill out the following student information form: https://goo.gl/forms/YBHa3R9fr66I2MV43.

2. A partition of a set A is defined as a collection of subsets $A_1, A_2, ..., A_n$ such that:

   - $A_i \subseteq A$
   - $A_i \cap A_j = \emptyset, \forall i \neq j$
   - $A_1 \cup A_2 \cup ... \cup A_n = A$

   For any other set $B \subset A$, show that

   $$|B| = |B \cap A_1| + |B \cap A_2| + ... + |B \cap A_n|$$

   You don't need to do anything rigorous — draw a picture, and justify to yourself why this is true. *As an aside: In mathematics, we sometimes use the $\coprod$ symbol to denote a union of disjoint sets. In our case, we could say $A_1 \coprod A_2 \coprod ... \coprod A_n = A$, since each A_i is disjoint from one another.*

3. In this question, we will introduce the Principle of Inclusion-Exclusion, which allows us to measure the size of the union of two sets. We will study this more when we learn counting, as there are significant implications of PIE in combinatorics.

   a. The Principle of Inclusion-Exclusion for two sets states that $|A \cup B| = |A| + |B| - |A \cap B|$. Derive this identity. *(Hint: Draw a picture.)*

   b. The Principle of Inclusion-Exclusion for three sets states that $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. Derive this identity.

   c. (Optional) Generalize the Principle of Inclusion-Exclusion for any number of n sets. *(Hint: It may help to first derive the expression for four sets. Do you notice a pattern?)*

4. Let A and B be sets. Determine $|A - B| + |B - A|$ (that is, the size of the set of elements that are either in A, or in B, but not both) in terms of $|A|, |B|$ and $|A \cap B|$.

5. The following information may be useful in the remaining questions.

   - $\mathbb{N} = \{1, 2, 3, 4, ...\}$

- $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$
- $\mathbb{R}_{\geq 0}$ = the set of all non-negative real numbers

Sets A, B, C are defined over a universe $\mathbb{U} = \{z : z \in \mathbb{N}_0, z \leq 25\}$ as follows:

- $A = \{x : x \text{ is prime}, x \leq 25\}$
- $B = \{2k : k \in \mathbb{N}_0, k \leq 12\}$
- $C = \{t^2 : t \in \mathbb{N}_0, t \leq 5\}$

Determine the sets that result after each of these set operations.

a. $A \cap B$

b. $A^C \cup B^C$ *(Hint: How can you use De Morgan's Laws to re-use your result from part a?)*

c. $(A \cup B) \cap C$

d. $B - C$

e. $A \setminus B^C$

f. $A^C \cap B^C \cap C^C$

6. In lecture, we showed that the composition of two injective functions is also injective, as follows:

   - Assume f, g are both one-to-one functions.
   - Consider $f(g(x_1)) = f(g(x_2))$. Since $f(\cdot)$ is injective, we have that $g(x_1) = g(x_2)$.
   - Since $g(\cdot)$ is injective, we have that $x_1 = x_2$.
   - Therefore, we have that $f(g(x_1)) = f(g(x_2))$ implies that $x_1 = x_2$, meaning that the function $f(g(x))$ is injective.

   Use a similar argument to show that the composition of two surjective functions is also surjective.

7. Determine whether each of the following functions is injective, surjective, both (bijective) or none.

   a. $f : \{2, 3, 4\} \to \{2, 3, 4\}, \{(2, 2), (3, 2), (4, 4)\}$

   b. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$

   c. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 + 3x^2 - 7x - 2$ *(Hint: Think about the derivative of $f(x)$, and the test for injections we saw in lecture).*

   d. $f : \mathbb{R}_{\geq 0} \to \mathbb{N}, f(x) = \lceil x \rceil$ *(Hint: This is the ceiling function.)*

   e. $f : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}, f(x, y) = x^2 + y^2$

   f. $f : \mathbb{N} \to \{t : t \in \mathbb{N}, t \text{ is prime}\}, f(x) = \text{the } x^{th} \text{ prime number}$