Problem Set 10: Final Review

CS 198-087: Introduction to Mathematical Thinking
UC Berkeley EECS
Fall 2018

This homework will not be collected. Instead, we intend it to be practice for the upcoming final. **This homework is not comprehensive; we highly encourage you to review material from before the midterm.**

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1. Prove that $\gcd(a, b) \cdot \text{lcm}(a, b) = a \cdot b$.
2. Determine the following inverses.
   a. 13^{-1} mod 33
   b. 15^{-1} mod 24
   c. 19^{-1} mod 90

   Use the modular exponentiation techniques we've seen in previous homeworks (FLT, extended FLT, repeated squaring) to evaluate the following quantities.
3. a. 18^{12} mod 26
   b. 9^{122} mod 143
   c. 8^{67} mod 15
   d. 10^{35} mod 17
4. Determine the following quantities.
   a. The number of subsets of $\{1, 2, 3, 4, ..., 50\}$ that are not subsets of $\{1, 2, 3, 4, ..., 10\}$ or $\{2, 4, 6, 8, ...48, 50\}$
   b. The number of multiples of 5, 7 or 12 that are less than or equal to $5^3 \cdot 7^3 \cdot 12^3$
   c. The number of factors of 1400 that are not multiples of $2^2 \cdot 7$

   Suppose I have 100 $1 dollar bills that I want to distribute between three of my friends, LeBron, Lonzo and Lance.

   How many ways can this be done...
5. a. In general, with no restrictions (other than that everyone receives some non-negative integer amount)?
   b. If everyone receives at least $1?

c. If everyone receives at least \$x, for $0 \leq x \leq 33$?

d. Such that LeBron and Lonzo receive the same amount? (*Hint: How can we format this as solving the number of solutions to $x + y = 50$?*)

e. Such that any two of them receive the same amount?

f. Such that LeBron receives at least \$x, and Lavar receives at most \$y?

6. Triangular numbers are numbers in the set $\{1, 3, 6, 10, 15, 21, ...\}$. The n-th triangular number, for $n \geq 1$, is given by $\binom{n+1}{2}$.

   a. Determine a closed form expression for

$$1 + 3 + 6 + 10 + ... + \binom{n+1}{2} = \sum_{k=2}^{n+1} \binom{k}{2}$$

   using the fact that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. It should be a cubic polynomial in n.

   b. Prove your closed form expression holds using induction.

7. a. Let $f(x) = 5x^3 - 4x^2 + 16x - 3$ have roots r_1, r_2, r_3. Find $r_1^2 r_2 r_3 + r_1 r_2^2 r_3 + r_1 r_2 r_3^2$.

   b. Find all values of m such that $2x^2 - mx - 8$ has roots that differ by $m - 1$.

   c. Suppose a and b satisfy $x^2 - mx + 2 = 0$. Also, suppose $a + \frac{1}{b}$ and $b + \frac{1}{a}$ satisfy $x^2 - px + q = 0$. Determine q in terms of a, b, p, m.

8. In each of the following expansions, find the coefficient of x^{13}.

   a. $(x^3 - \frac{1}{x})^7$

   b. $(x^5 - 1)^6 (2x^2 + 3x)^3$

   Let's compare decimal approximations using both the Binomial Theorem and a Taylor Series approximation. Suppose we want to estimate $\sqrt{37}$.

9. a. Approximate $\sqrt{37}$ by finding the first three terms of the Taylor Series approximation of $f(x)$ centered around $a = 36$, letting $x = 1$.

   b. Approximate $\sqrt{37}$ by expanding the first three terms of the binomial expansion of $(36 + 1)^{1/2}$.

   c. What do you notice?

10. Determine the polynomial that interpolates $S = \{(1, 4), (2, 6), (5, 3)\}$ under

    a. mod 7

    b. mod 11