Problem Set 10: Final Review

CS 198-087: Introduction to Mathematical Thinking
UC Berkeley EECS
Fall 2018

This homework will not be collected. Instead, we intend it to be practice for the upcoming final. **This homework is not comprehensive; we highly encourage you to review material from before the midterm.**

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1. Prove that $\gcd(a, b) \cdot \text{lcm}(a, b) = a \cdot b$.

Solution:

$$\begin{aligned}
\gcd(a,b) \cdot \text{lcm}(a,b) &= (p_1^{min(a_1,b_1)} \cdot p_2^{min(a_2,b_2)} \cdot \ldots \cdot p_k^{min(a_k,b_k)}) \cdot (p_1^{max(a_1,b_1)} \cdot p_2^{max(a_2,b_2)} \cdot \ldots \cdot p_k^{max(a_k,b_k)}) \\
&= p_1^{max(a_1,b_1)+min(a_1,b_1)} \cdot p_2^{max(a_2,b_2)+min(a_2,b_2)} \cdot \ldots \cdot p_k^{max(a_k,b_k)+min(a_k,b_k)} \\
&= p_1^{a_1+b_1} \cdot p_2^{a_2+b_2} \cdot \ldots \cdot p_k^{a_k+b_k} \\
&= (p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}) \cdot (p_1^{b_1} \cdot p_2^{b_2} \cdot \ldots \cdot p_k^{b_k}) \\
&= ab
\end{aligned}$$

2. Determine the following inverses.

   a. $13^{-1} \bmod 33$

   b. $15^{-1} \bmod 24$

   c. $19^{-1} \bmod 90$

Solution:

a. The calls we'd make to the Euclidean algorithm are $(33, 13)$, $(13, 7)$, $(7, 6)$ and $(6, 1)$. We can then write the following relationships using the division algorithm:

$$\begin{aligned}
33 &= 2 \cdot 13 + 7 \\
13 &= 1 \cdot 7 + 6 \\
7 &= 1 \cdot 6 + 1
\end{aligned}$$

Rearranging for the remainders, we have

$$7 = 33 - 2 \cdot 13$$
$$6 = 13 - 1 \cdot 7$$
$$1 = 7 - 1 \cdot 6$$

Substituting, we have

$$\begin{aligned} 1 &= 7 - 1 \cdot 6 \\ &= 7 - 1 \cdot (13 - 1 \cdot 7) = 2 \cdot 7 - 13 \\ &= 2 \cdot (33 - 2 \cdot 13) - 13 \\ &= 2 \cdot 33 - 5 \cdot 13 \end{aligned}$$

Therefore, -5, or $\boxed{28}$ (since $-5 + 33 = 28$) is the inverse of 13 in mod 33.

b. This inverse does not exist, as $\gcd(15, 24) = 3 \neq 1$.

c. The process is identical to that in part a, and so the steps are not reproduced below. However, you should ensure that you get the result $\boxed{19}$. To verify, $19 \cdot 19 = 361 = 90 \cdot 4 + 1$.

3. Use the modular exponentiation techniques we've seen in previous homeworks (FLT, extended FLT, repeated squaring) to evaluate the following quantities.

   a. 18^{12} mod 26

   b. 9^{122} mod 143

   c. 8^{67} mod 15

   d. 10^{35} mod 17

Solution: We will heavily use the fact that $a^{(p-1)(q-1)} \equiv 1 \bmod (p-1)(q-1)$ for relatively prime p, q. We've referred to this as "extended FLT."

a. We know $26 = 2 \cdot 13$, both of which are prime. Thus, $a^{(2-1)(13-1)} \equiv a^{12} \equiv 1 \bmod 26$. Therefore, $18^{12} \equiv \boxed{1}$ mod 26.

b. Again, we can factor 143 as $11 \cdot 13$. $(11 - 1) \cdot (13 - 1) = 120$, telling us that $a^{120} \equiv 1 \bmod 120$. Then, $9^{122} \equiv 9^{120} \cdot 9^2 \equiv 9^2 \equiv \boxed{81}$ mod 120.

c. Note: parts c and d are very similar. We will do part c using repeated squaring, and d using Fermat's Little Theorem.

   We can write 67 as the sum of powers of two, as $67 = 64 + 2 + 1$. Once we find expressions for $8, 8^2$ and 8^{64}, we can multiply them together to find our result (in mod 15, of course).

$$8^1 \equiv 8$$
$$8^2 \equiv 64 \equiv 4$$
$$8^4 \equiv 4^2 \equiv 16 \equiv 1$$
$$8^16 \equiv (8^4)^4 \equiv 1^4 \equiv 1$$

Then, $8^{67} \equiv 8^{64} \cdot 8^2 \cdot 8^1 \equiv 1 \cdot 4 \cdot 8 \equiv 32 \equiv \boxed{2} \pmod{15}$.

d. Using Fermat's Little Theorem, we have that $a^{16} \equiv 1 \pmod{17}$. Thus, $10^{16} \equiv 1$. Then,

$$\begin{aligned} 10^{35} &\equiv 10^{32} \cdot 10^3 \\ &\equiv (10^{16})^2 \cdot 10^3 \equiv 10^3 \\ &\equiv 10^2 \cdot 10 \equiv (-2) \cdot 10 \\ &\equiv -20 \equiv \boxed{14} \pmod{17} \end{aligned}$$

4. Determine the following quantities.

a. The number of subsets of $\{1, 2, 3, 4, ..., 50\}$ that are not subsets of $\{1, 2, 3, 4, ..., 10\}$ or $\{2, 4, 6, 8, ...48, 50\}$

b. The number of multiples of 5, 7 or 12 that are less than or equal to $5^3 \cdot 7^3 \cdot 12^3$

c. The number of factors of 1400 that are not multiples of $2^2 \cdot 7$

Solution:

a. Let $\mathbb{U} = \{1, 2, 3, 4, ..., 50\}$, $A = \{1, 2, 3, 4, ..., 10\}$ and $B = \{2, 4, 6, 8, ..., 48, 50\}$.

We will proceed by finding the number of subsets of either A or B. Recall, the power set of S is the set of all subsets of S.

$$|P(A) = 2^{10}|$$
$$|P(B) = 2^{25}|$$

The intersection of the two sets is $A \cap B = \{2, 4, 6, 8, 10\}$, and $|P(A \cap B)| = 2^5$. Therefore, the number of subsets of A or B is $2^{10} + 2^{25} - 2^5$, and so the number of subsets of $\mathbb{U}$ that are not subsets of A or B is $\boxed{2^{50} - 2^{10} - 2^{25} + 2^5}$.

b. Let M_i represent the set of multiples of i less than $5^3 \cdot 7^3 \cdot 12^3$.

As a smaller example, consider $5 \cdot 12$. There are 12 multiples of 5 less than 60: $5 \cdot 1, 5 \cdot 2, ...5 \cdot 12$. We can generalize this to say there are $\frac{5^3 \cdot 7^3 \cdot 12^3}{i}$ multiples of i less than

$5^3 \cdot 7^3 \cdot 12^3$.

$$|M_5 \cup M_7 \cup M_{12} = |M_5| + |M_7| + |M_{12}| - |M_5 \cap M_7| - |M_5 \cap M_{12}| - |M_7 \cap M_{12}| + |M_5 \cap M_7 \cap M_{12}|$$
$$= \boxed{5^2 7^3 12^3 + 5^3 7^2 12^3 + 5^3 7^3 12^2 - 5^2 7^2 12^3 - 5^2 7^3 12^2 - 5^3 7^2 12^2 + 5^2 7^2 12^2}$$

c. We know that (# factors of 1400, not multiples of $2^2 \cdot 7$) is equal to (# factors of 1400) minus (# factors of 1400, multiples of $2^2 \cdot 7$).

1400 prime factors as $2^3 \cdot 5^2 \cdot 7$, meaning it has $4 \cdot 3 \cdot 2 = 24$ factors. To find the number of factors that are multiples of $2^2 \cdot 7$, our number of options for each exponent now decrease. Now, there are only 2 options for the exponent on 2 (2 or 3), still 3 for the exponent on 5 (0, 1, or 2) and 1 for the exponent on 7 (must be 1). This gives us $2 \cdot 3 = 6$ factors of 1400 that are multiples of $2^2 \cdot 7$. Then, the number of factors that are not multiples of $2^2 \cdot 7$ are $24 - 6 = \boxed{18}$.

5. Suppose I have 100 $1 dollar bills that I want to distribute between three of my friends, LeBron, Lonzo and Lance.

How many ways can this be done...

a. In general, with no restrictions (other than that everyone receives some non-negative integer amount)?

b. If everyone receives at least $1?

c. If everyone receives at least $t, for $0 \leq x \leq 33$?

d. Such that LeBron and Lonzo receive the same amount? (*Hint: How can we format this as solving the number of solutions to $x + y = 50$?*)

e. Such that any two of them receive the same amount?

f. Such that LeBron receives at least $t, and Lance receives at most $y?

Solution: We will model each question as finding the number of non-negative integer solutions to $x_1 + x_2 + x_3 = 100$, with different sets of constraints in each. Let x_1 represent LeBron, x_2 Lonzo and x_3 Lance.

a. Here, our only constraints are $x_1, x_2, x_3 \geq 0$. This is given by the standard stars-and-bars solution of $\binom{100+2}{2} = \boxed{\binom{102}{2}}$, since we have 100 stars and 2 bars.

b. Defining $x_i' = x_i - 1$ gives us $x_1' + x_2' + x_3' = 97$, which has $\binom{97+2}{2} = \boxed{\binom{99}{2}}$ solutions.

c. Now, we define $x_i' = x_i - t$, for $0 \leq x \leq 33$. Then:

$$\begin{aligned} x_1 + x_2 + x_3 &= 100 \\ (x_1 - t) + (x_2 - t) + (x_3 - t) &= 100 - 3t \\ x_1' + x_2' + x_3' &= 100 - 3t \end{aligned}$$

which has $\boxed{\binom{100 - 3x + 2}{2}}$ solutions.

d. Now, we set $x_1 = x_2$, meaning we are looking at $2x_1 + x_3 = 100$. Since $2x_1$ is an even number, and 100 is even, we know that x_3 must also be even. So, we set $x_3 = 2k$, for some integer k, where $0 \leq k \leq 50$. We are now looking at the number of non-negative integer solutions to $x_1 + k = 50$, which can be modelled using 50 stars and 1 bar. This has $\binom{50+1}{1} = \boxed{51}$ solutions.

e. Now, we consider three cases, $x_1 = x_2$, $x_1 = x_3$ and $x_2 = x_3$. Note, we don't need to consider any overlap, because it's impossible for $x_1 = x_2 = x_3$, as $3x_1 = 100$ has no integer solutions!

   Then, our answer is just three times the answer in the previous part, meaning this situation has $3 \cdot 51 = \boxed{153}$ solutions.

f. First, we deal with the constraint that $x_1 \leq t$. By defining $x_1' = x_1 - t$, we are now looking at the number of solutions to $x_1' + x_2 + x_3 = 100 - t$, where each variable can be a non-negative integer.

   Now, looking at the constraint $x_3 \leq y$, we can break this up into $y + 1$ separate cases: either $x_3 = 0$, or $x_3 = 1$, or $x_3 = 2$, ..., or $x_3 = y$.

   When $x_3 = 0$, we are now looking at the number of solutions to $x_1' + x_2 = 100 - t$, which is $\binom{100-t+1}{1}$. If $x_3 = 1$, this quantity is now $\binom{100-t-1+1}{1}$. In general, if $x_3 = i$, we are finding the number of solutions to $x_1' + x_2 = 100 - t - i$, which is given by $\binom{100-t-i+1}{1}$. We now need to sum from $i = 0$ to $i = y$, as these represent all the possible values of x_3.

   Also, notice that $\binom{x}{1} = x$.

$$\begin{aligned} \sum_{i=0}^{y} \binom{100 - t - i + 1}{1} &= \sum_{i=0}^{y} (100 - t - i + 1) \\ &= \sum_{i=0}^{y} (101 - t) - \sum_{i=0}^{y} i \\ &= \boxed{(101 - t)(y + 1) - \frac{y(y + 1)}{2}} \end{aligned}$$

6. Triangular numbers are numbers in the set $\{1, 3, 6, 10, 15, 21, ...\}$. The n-th triangular number, for $n \geq 1$, is given by $\binom{n+1}{2}$.

   a. Determine a closed form expression for

$$1 + 3 + 6 + 10 + ... + \binom{n+1}{2} = \sum_{k=2}^{n+1} \binom{k}{2}$$

   using the fact that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. It should be a cubic polynomial in n.

   b. Prove your closed form expression holds using induction.

Solution:

a.

$$\begin{aligned}
\sum_{k=2}^{n+1} \binom{k}{2} &= \sum_{k=2}^{n+1} \frac{k(k-1)}{2} \\
&= \frac{1}{2}\left(\sum_{k=2}^{n+1} k^2 - \sum_{k=2}^{n+1} k\right) \\
&= \frac{1}{2}\left(\left(\sum_{k=1}^{n+1} k^2 - 1^2\right) - \left(\sum_{k=1}^{n+1} k - 1\right)\right) \\
&= \frac{1}{2}\left(\frac{(n+1)(n+2)(2n+3)}{6} - 1 - \frac{(n+1)(n+2)}{2} + 1\right) \\
&= \frac{1}{2}\left(\frac{2n(n+1)(n+2)}{6}\right) = \boxed{\frac{n(n+1)(n+2)}{6}}
\end{aligned}$$

b. *Base Case:* $n = 1$

$1 = \frac{1(2)(3)}{6}$, therefore the base case holds.

Induction Hypothesis: Assume $n = j$ holds

Assume $\sum_{k=2}^{j+1} \binom{k}{2} = \frac{j(j+1)(j+2)}{6}$ for some arbitrary integer j.

Induction Step: Prove $n = j + 1$ holds

$$\sum_{k=2}^{j+2} \binom{k}{2} = \sum_{k=2}^{j+1} \binom{k}{2} + \binom{j+2}{2}$$
$$= \frac{j(j+1)(j+2)}{6} + \frac{3(j+2)(j+1)}{2 \cdot 3}$$
$$= \boxed{\frac{(j+1)(j+2)(j+3)}{6}}$$

Therefore, by induction, this expression holds.

7. a. Let $f(x) = 5x^3 - 4x^2 + 16x - 3$ have roots r_1, r_2, r_3. Find $r_1^2 r_2 r_3 + r_1 r_2^2 r_3 + r_1 r_2 r_3^2$.

   b. Find all values of m such that $2x^2 - mx - 8$ has roots that differ by $m - 1$.

   c. Suppose a and b satisfy $x^2 - mx + 2 = 0$. Also, suppose $a + \frac{1}{b}$ and $b + \frac{1}{a}$ satisfy $x^2 - px + q = 0$. Determine q in terms of a, b, p, m.

Solution:

a. We can factor $r_1^2 r_2 r_3 + r_1 r_2^2 r_3 + r_1 r_2 r_3^2$ as $r_1 r_2 r_3 (r_1 + r_2 + r_3)$. Then, from Vieta's, we know that $r_1 + r_2 + r_3 = -\frac{-4}{5} = \frac{4}{5}$ and $r_1 r_2 r_3 = -\frac{-3}{5} = \frac{3}{5}$. Then, the quantity we're looking for is $\boxed{\frac{12}{25}}$.

b. Suppose r_1, r_2 are the roots of this equation, and let's assume $r_1 \geq r_2$ (we could equivalently say $r_1 \geq r_2$ but it doesn't really matter).

Then, since we have the equation $2x^2 - mx - 8$, we know that $r_1 + r_2 = -\frac{-m}{2} = \frac{m}{2}$, $r_1 r_2 = -4$, and we want $r_1 - r_2 = m - 1$. Solving using the first and third equations, we can find the following expressions for r_1, r_2 in terms of m:

$$r_1 = \frac{3}{4}m - 1$$
$$r_2 = \frac{1}{2}m - r_1 = -\frac{1}{4}m + \frac{1}{2}$$

Then, since we have that $r_1 r_2 = -4$, we can multiply our expressions for r_1, r_2 and solve for m.

$$\begin{aligned} r_1 r_2 &= -4 \\ \left(\frac{3}{4}m - 1\right)\left(-\frac{1}{4}m + \frac{1}{2}\right) &= -4 \\ (3m-2)(m-2) &= 64 \\ 3m^2 - 8m - 60 = (m-6)(3m+10) &= 0 \end{aligned}$$

This tells us that the possible values for m are $\boxed{6, -\frac{10}{3}}$.

c. Since a, b are roots of $x^2 - mx + 2$, we know that $a + b = m$ and $ab = 2$.

Since $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are roots of $x^2 - px + q$, we know that $a + \frac{1}{b} + b + \frac{1}{a} = p$ and $\left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) = q$.

Expanding out the expression for q:

$$q = \left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) ab + 1 + 1 + \frac{1}{ab}$$

Since we know that $ab = 2$, we can actually determine a numerical value for q:

$$q = 2 + 1 + 1 + \frac{1}{2} = \boxed{\frac{9}{2}}$$

8. In each of the following expansions, find the coefficient of x^{13}.

   a. $(x^3 - \frac{1}{x})^7$

   b. $(x^5 - 1)^6(2x^2 + 3x)^3$

Solution:

a. First, we find the general term:

$$\begin{aligned} t_k &= \binom{7}{k} x^{3(7-k)} (-x^{-1})^k \\ &= (-1)^k \binom{7}{k} x^{21-4k} \end{aligned}$$

Setting $21 - 4k = 13$ gives us $k = 2$. Then,

$$t_2 = (-1)^2 \binom{7}{2} x^{21-8} = \binom{7}{2} x^{13} = \boxed{21} x^{13}$$

b. We find the general terms of both separate polynomials first. We can use the variable i for $(x^5 - 1)^6$ and j for $(2x^2 + 3x)^3$.

$$\begin{aligned} t_i &= \binom{6}{i} (x^5)^{6-i} (-1)^i \\ &= (-1)^i \binom{6}{i} x^{30-5i} \\ t_j &= \binom{3}{j} (2x^2)^{3-j} (3x)^j \\ &= \binom{3}{j} 2^{3-j} 3^j x^{6-j} \end{aligned}$$

Multiplying the two general terms together yields

$$t_{i,j} = (-1)^i \binom{6}{i} \binom{3}{j} 2^{3-j} 3^j x^{36-5i-j}$$

Now, we set the exponent $36-5i-j$ equal to 13, which simplifies to $5i+j = 13$, where $0 \leq i \leq 6$ and $0 \leq j \leq 3$. Plugging in $j = 0, j = 1, j = 2$ yields non-integer solutions for i, which do not make sense in this case (as i, j represent indices). Plugging in $j = 3$ yields $i = 4$. Then,

$$t_{i=4,j=3} = (-1)^4 \binom{6}{4} \binom{3}{3} 2^{3-3} 3^3 x^{36-5\cdot4-3} = 15 \cdot 27 x^{13} = \boxed{405} x^{13}$$

9. Let's compare decimal approximations using both the Binomial Theorem and a Taylor Series approximation. Suppose we want to estimate $\sqrt{37}$.

   a. Approximate $\sqrt{37}$ by finding the first three terms of the Taylor Series approximation of $f(x)$ centered around $a = 36$, letting $x = 1$.

   b. Approximate $\sqrt{37}$ by expanding the first three terms of the binomial expansion of $(36 + 1)^{1/2}$.

   c. What do you notice?

Solution:

a. If $f(x) = x^{1/2}$, then $f'(x) = \frac{1}{2} x^{-\frac{1}{2}}$ and $f''(x) = -\frac{1}{4} x^{-\frac{3}{2}}$.

Then,

$$
\begin{aligned}
f(36+x) &= f(36) + xf'(36) + \frac{x^2 f''(36)}{2!} \\
f(37) &= f(36) + f'(36) + \frac{f''(36)}{2!} \\
&= 6 + \frac{1}{2} \cdot \frac{1}{6} - \frac{1}{8 \cdot 216} \\
&= \boxed{6.08275}
\end{aligned}
$$

b. Recall, $\binom{n}{1} = n$ and $\binom{n}{2} = \frac{n(n-1)}{2}$.

$$
\begin{aligned}
(36+1)^n &= 36^n + n \cdot 36^{n-1} + \frac{n(n-1)}{2} 36^{n-2} \\
(36+1)^{\frac{1}{2}} &= 36^{\frac{1}{2}} + \frac{1}{2} \cdot 36^{-\frac{1}{2}} + \frac{(\frac{1}{2})(-\frac{1}{2})}{2} 36^{-\frac{3}{2}} \\
&= 6 + \frac{1}{2} \cdot \frac{1}{6} - \frac{1}{8 \cdot 216} \\
&= \boxed{6.08275}
\end{aligned}
$$

c. In this case, they both happened to be be the same!

10. Determine the polynomial that interpolates $S = \{(1,4),(2,6),(5,3)\}$ under

a. mod 7

b. mod 11

Solution:

a. First, we make sub-polynomials $p_1(x)$, $p_2(x)$ and $p_3(x)$.

Recall, we are trying to find this polynomial modulo 7. We will make simplifications in modulo 7 as we go in order to make the manual arithmetic easier. Some of these simplifying steps are rather arbitrary, and could be saved until the end. Here, $x_1 = 1$, $x_2 = 2$ and $x_3 = 5$.

$$
\begin{aligned}
p_1(x) &= \frac{(x-2)(x-5)}{(1-2)(1-5)} = \frac{x^2-7x+10}{4} \\
&\equiv \frac{x^2+3}{4} \\
&\equiv (4)^{-1}(x^2+3) \\
&\equiv 2(x^2+3)
\end{aligned}
$$

In the above, we used the fact that $-7x \equiv 0$ mod 7.

$$
\begin{aligned}
p_2(x) &= \frac{(x-1)(x-5)}{(2-1)(2-5)} = \frac{x^2-6x+5}{-3} \\
&\equiv \frac{x^2+x-2}{4} \\
&\equiv 2(x^2+x-2)
\end{aligned}
$$

$$
\begin{aligned}
p_3(x) &= \frac{(x-1)(x-2)}{(5-1)(5-2)} = \frac{x^2-3x+2}{12} \\
&\equiv \frac{x^2-3x+2}{5} \\
&\equiv 3(x^2-3x+2)
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
p(x) &= y_1p_1(x) + y_2p_2(x) + y_3p_3(x) \\
&= 4 \cdot 2(x^2+3) + 6 \cdot 2(x^2+x-2) + 3 \cdot 3(x^2-3x+2) \\
&\equiv x^2+3-2(x^2+x-2)+2(x^2-3x+2) \\
&\equiv x^2+3-2x^2-2x+4+2x^2-6x+4 \\
&\equiv x^2-8x+11 \\
&\equiv \boxed{x^2-x+4} \text{ (mod 7)}
\end{aligned}
$$

As a sanity check, we can verify that if $p(x) = x^2 - x + 4$, then $p(1) \equiv 4, p(2) \equiv 6$ and $p(5) \equiv 3$, all in mod 7.

b. We will follow the same process, but instead make simplifications in modulo 11.

$$p_1(x) = \frac{(x-2)(x-5)}{(1-2)(1-5)} = \frac{x^2-7x+10}{4}$$
$$\equiv 3(x^2+4x-1)$$

$$p_2(x) = \frac{(x-1)(x-5)}{(2-1)(2-5)} = \frac{x^2-6x+5}{-3}$$
$$\equiv 7(x^2-6x+5)$$

$$p_3(x) = \frac{(x-1)(x-2)}{(5-1)(5-2)} = \frac{x^2-3x+2}{12}$$
$$\equiv x^2-3x+2$$

Then,

$$\begin{aligned} p(x) &= y_1p_1(x) + y_2p_2(x) + y_3p_3(x) \\ &= 4 \cdot 3(x^2+4x-1) + 6 \cdot 7(x^2-6x+5) + 3(x^2-3x+2) \\ &\equiv x^2+4x-1-2x^2+12x-10+3x^2-9x+6 \\ &\equiv \boxed{2x^2+7x-5} \pmod{11} \end{aligned}$$

Again, we can verify that if $p(x) = 2x^2+7x-5$, then $p(1) \equiv 4, p(2) \equiv 6$ and $p(5) \equiv 3$, all in mod 11.