Problem Set 1: Set Theory, Functions

CS 198-087: Introduction to Mathematical Thinking
UC Berkeley EECS
Fall 2018

This homework is due on Wednesday, September 12th, at 6:30PM, on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. Use LaTeX if possible.

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1. Fill out the following student information form: https://goo.gl/forms/Kv5og6iAcKrO7jBj1.
2. Revisit the diagnostic for this course at http://imt-decal.org/assets/diagnostic.pdf.
   a. Which of the problems are you able to comfortably answer?
   b. Which problem did you find to be the easiest? The most difficult?
3. Let $A_1, A_2, ...A_n$ be disjoint sets such that make up the universe. That is, $\cup_{i=1}^{n} A_i = A_1 \cup A_2 \cup ... \cup A_n = \mathbb{U}$. For any other set $B \subset \mathbb{U}$, show that

   $$|B| = |B \cap A_1| + |B \cap A_2| + ... + |B \cap A_n|$$

   This identity is used in deriving the total probability rule in probability theory. *(Hint: Draw a picture.)*

   Solution: Since $A_1 \cup A_2 \cup ... \cup A_n = \mathbb{U}$, and that all A_i are disjoint, we know that B is divided into n parts, each of which overlaps with exactly one A_i. As a result, we have that $|B| = |B \cap A_1| + |B \cap A_2| + ... + |B \cap A_n|$.

4. In this question, we will introduce the Principle of Inclusion-Exclusion, which allows us to measure the size of the union of two sets. We will study this more when we learn counting, as there are significant implications of PIE in combinatorics.
   a. The Principle of Inclusion-Exclusion for two sets states that $|A \cup B| = |A| + |B| - |A \cap B|$. Derive this identity. *(Hint: Draw a picture.)*
   b. The Principle of Inclusion-Exclusion for three sets states that $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. Derive this identity.
   c. (Optional) Generalize the Principle of Inclusion-Exclusion for any number of n sets. *(Hint: It may help to first derive the expression for four sets. Do you notice a pattern?)*

Solution:

a. First, we count every item in A and B individually, yielding $|A| + |B|$. We then see that the intersection $A \cap B$ has been counted twice once in $|A|$, and once in $|B|$. By subtracting $|A \cap B|$ we yield $|A \cup B| = |A| + |B| - |A \cap B|$ as required. (Derived in lecture, see video in textbook.)

b. Again, we start by counting each set individually, giving us $|A| + |B| + |C|$. We now notice that each pairwise overlap has been counted twice $|A \cap B|$ was counted in both $|A|$ and $|B|$, $|A \cap C|$ was counted in both $|A|$ and $|C|$, and $|B \cap C|$ was counted in both $|B|$ and $|C|$; additionally, the triple intersection $|A \cap B \cap C|$ is counted three times. By subtracting $|A \cap B|$, $|A \cap C|$ and $|B \cap C|$, we have subtracted the triple overlap $|A \cap B \cap C|$ three times (as it is part of each pairwise intersection). Since it was originally counted three times, we need to add it back once. Thus, our final relation yields $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

c. In general, we sum all individual cardinalities, subtract pairwise intersections, add back intersections of triplets, subtract intersections of each combination of four sets, and so on and so forth.

5. Let A, B and C be sets.

a. Determine $|A - B| + |B - A|$ (that is, the size of the set of elements that are either in A, or in B, but not both) in terms of $|A|, |B|$ and $|A \cap B|$.

b. Determine $|(A - B) - C| + |(B - A) - C| + |(C - A) - B|$, the size of the set of elements that are in exactly one of A, B, C in terms of the relevant quantities. *Hint: If A, B and C are disjoint, what is this quantity? Make sure your expression satisfies this case as well.*

Solution:

a. By the principle of inclusion-exclusion, we know the number of elements in A or B (including the overlap) is $|A| + |B| - |A \cap B|$. However, we want to exclude the elements that are in the overlap, so we subtract by $|A \cap B|$ again, yielding $|A| + |B| - 2|A \cap B|$.

b. One again, we start off with the principle of inclusion-exclusion, which tells us that $|A \cup B| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. Now, we want to remove everything that is in an intersection. A good way to proceed would be removing everything in $A \cap B$, $A \cap C$ and $B \cap C$.
In doing this, we removed the triple overlap $A \cap B \cap C$ twice times, so we need to add it back twice. We can summarize our changes to the formula given by inclusion-exclusion as $\Delta = -|A \cap B| - |A \cap C| - |B \cap C| + 2|A \cap B \cap C|$, and so our final result is

$$|A| + |B| + |C| - 2|A \cap B| - 2|A \cap C| - 2|B \cap C| + 3|A \cap B \cap C|$$

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6. Show that De Morgan's Law for sets holds; that is, verify the following:

$$(A \cap B)^C = A^C \cup B^C$$

(Hint: Consider some universe, and two sets A, B that overlap. Assign names to different subsets, and show that both sides of the equals sign count the same objects.)

Solution: We can show that two sets are equal by showing that they're subsets of one another. First, we'll show that $(A \cap B)^C \subseteq A^C \cup B^C$. If $x \in (A \cap B)^C$, then $x \notin (A \cap B)$. If x is not in the intersection of A and B, then either $x \notin A$ or $x \notin B$, meaning that $x \in A^C$ or $x \in B^C$. This means, $x \in A^C \cup B^C$. We have that $x \in (A \cap B)^C \implies x \in A^C \cup B^C$, therefore $(A \cap B)^C \subseteq A^C \cup B^C$. Let's look at the opposite direction. (Recall, $\vee$, the disjunction symbol, translates to "or".)

$$\begin{aligned} x \in A^C \cup B^C &\implies x \in A^C \vee x \in B^C \\ &\implies x \notin A \vee x \notin B \\ &\implies x \notin A \cap B \\ &\implies x \in (A \cap B)^C \end{aligned}$$

Since we've shown that $(A \cap B)^C \subseteq A^C \cup B^C$ and $A^C \cup B^C \subseteq (A \cap B)^C$, we have that $(A \cap B)^C = A^C \cup B^C$. *As an extra exercise, try showing the other version of De Morgan's Law: that $(A \cup B)^C = A^C \cap B^C$.*

7. In lecture, we showed that the composition of two injective functions is also injective, as follows:

Assume f, g are both one-to-one functions. Consider $f(g(x_1)) = f(g(x_2))$. Since $f(\cdot)$ is injective, we have that $g(x_1) = g(x_2)$. Since $g(\cdot)$ is injective, we have that $x_1 = x_2$. Therefore, we have that $f(g(x_1)) = f(g(x_2))$ implies that $x_1 = x_2$, meaning that the function $f(g(x))$ is injective.

Use a similar argument to show that the composition of two surjective functions is also surjective.

Solution: Suppose $g : A \to B$ and $f : B \to C$. Notice this means that $f(g(\cdot)) : A \to C$. To prove that the composition of two injective functions is injective, we showed that $f(g(x_1)) = f(g(x_2)) \implies x_1 = x_2$. To show that the composition of two surjective functions is also surjective, we need to show that $\forall c \in C, \exists\, a \in A : f(g(a)) = c$.

Since $f : B \to C$ is surjective, $\forall c \in C, \exists\, b \in B : f(b) = c$.

Then, since $g : A \to B$ is surjective, $\forall b \in B, \exists\, a \in A : g(a) = b$.

Putting these two statements together, we have that $\forall a \in A, \exists\, c \in C : f(g(a)) = c$.

$\forall$ translates to "for all", and $\exists$ translates to "there exists". Refer to the textbook if you are unsure of the notation.

8. Use set-builder notation to describe each of the following sets. *(Hint: You may find the following definition handy: $\mathbb{N}_0 = \{0, 1, 2, 3, 4, ...\}$. Hint 2: You can specify multiple conditions when using set-builder notation.)*

   a. $\{0, 2, 4, 6, 8, 10, 12, ...\}$

   b. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots \frac{1}{13}\}$

   c. $\{0, 1, 00, 01, 100, 101, 110, 111, 1000, ...\}$

Solution:

a $\{2n : n \in \mathbb{N}_0\}$

b $\{\frac{1}{n} : n \in \mathbb{N}\}$

c $\{\text{binary}(n) : n \in \mathbb{N}_0\}$

9. Sets A, B, C are defined over a universe $\mathbb{U} = \{z : z \in \mathbb{N}_0, z \leq 25\}$ as follows:

   - $A = \{x : x \text{ is prime}, x \leq 25\}$
   - $B = \{2k : k \in \mathbb{N}_0, k \leq 25\}$
   - $C = \{t^2 : t \in \mathbb{N}_0, t \leq 25\}$

   Determine the sets that result after each of these set operations. *(Hint: A set with one element is still a set.)*

   a. $A \cap B$

   b. $(A \cup B) \cap C$

   c. $B - C$

   d. $A \setminus B^C$

   e. $A^C \cap B^C \cap C^C$

Solution:

a. $\{2\}$

b. $\{0, 4, 16\}$

c. $\{2, 6, 8, 10, 12, 14, 18, 20, 22, 24\}$

d. $\{2\}$

e. $\{5, 15, 21\}$

10. As we will see in Section 1.3, we have the following definitions:

   - $\mathbb{N} = \{1, 2, 3, 4, ...\}$

- $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$
- $\mathbb{R}_{\geq 0}$ = the set of all non-negative real numbers

Determine whether each of the following functions is injective, surjective, both (bijective) or none.

a. $f : \{2, 3, 4\} \to \{2, 3, 4\}, \{(2, 2), (3, 2), (4, 4)\}$

b. $f : \{2, 3, 4\} \to \{2, 3, 4\}, \{(2, 3), (3, 2), (4, 4)\}$

c. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$

d. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 + 3x^2 - 7x - 2$ *(Hint: Think about what the graph of f looks like.)*

e. $f : \mathbb{R}_{\geq 0} \to \mathbb{N}, f(x) = \lceil x \rceil$ *(Hint: This is the ceiling function.)*

f. $f : \mathbb{R}_{\geq 0} \to \mathbb{N}, f(x) = \lfloor x \rfloor$ *(Hint: This is the floor function.)*

g. $f : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}, f(x, y) = x^2 + y^2$

h. $f : \mathbb{N} \to \{t : t \in \mathbb{N}, t \text{ is prime}\}, f(x) = \text{the } x^{th} \text{ prime number}$

Solution:

a. f is neither injective nor surjective. f is not injective, as $f(2) = f(3) = 2$. f is not surjective as 3 has no pre-image (that is, there is no x such that $f(x) = 3$).

b. f is a bijection. f is injective, as $f(x)$ is unique $\forall x$. f is injective, as Range$\{x\}$ = Codomain$\{x\}$.

c. f is a bijection. f is injective, as $a^3 = b^3 \implies a = b, \forall a, b \in \mathbb{R}$ (note, if we allow the domain to span the set of complex numbers, this is no longer true.) f is surjective since $\forall y \in \mathbb{R}, \exists\, x \in \mathbb{R} : y = x^3$. Specifically, every real number has a cubed root.

d. f is a surjection, but not an injection. f is not an injection, as it has one local maximum and one local minimum, implying that it changes directions twice, meaning it does not pass the horizontal line test required for functions to be injections. f is a surjection as there is always a solution to $f(x) = c, \forall c \in \mathbb{R}$. (Think about how $f(x)$ extends infinitely in the positive and negative directions.)

e. f is a surjection, but not an injection. f is not an injection as $\lceil 1.5 \rceil = \lceil 1.6 \rceil = 2$, but $1.5 \neq 1.6$. f is surjective, as for every positive integer n, there exists at least one a such that $\lceil a \rceil = n$. For example, $\lceil n \rceil = n$.

f. f is neither an injection nor surjection. f is not an injection since, for example, $\lfloor 1.5 \rfloor = \lfloor 1.6 \rfloor = 1$, but $1.5 \neq 1.6$. f is a surjection, as for every positive integer n, there exists at least one a such that $\lfloor a \rfloor = n$. For example, $\lfloor n \rfloor = n$.

g. f is a surjection, but not an injection. f is not an injection, as $f(0, 5) = f(5, 0) = 25$, but $(0, 5) \neq (5, 0)$. f is a surjection, as for all non-negative real numbers r, there exists at least one ordered pair (x, y) such that $x^2 + y^2 = r$. One such ordered pair is $(\sqrt{r}, 0)$. *Note: f is a function, of multiple variables.*

h. f is a bijection. f is an injection, as the i^{th} prime number and j^{th} prime number are different, by nature, for all positive integers i, j. f is a surjection by definition our codomain is the set of all prime numbers, and the outputs of $f(\cdot)$ are precisely the prime numbers, in order $f(1)$ is the first prime number, $f(2)$ is the second prime number, and so on.