PROBLEM SET 2: NUMBER SETS, PROPOSITIONAL LOGIC

CS 198-087: INTRODUCTION TO MATHEMATICAL THINKING
UC BERKELEY EECS
SPRING 2019

This homework is due on Friday, February 15, 11:59 PM, on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX.

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1. Determine a bijection $f : A \to B$ between each pair of sets, and prove that f is a bijection. *(While this can be done by showing that such a mapping is invertible, and therefore bijective, the standard way we'll proceed in this class is by showing a function is both injective and surjective.)*

   a. $A = \{1, 2, 3, 4, 5, 6, ...\}, B = \{4, 7, 10, 13, 16, 19, ...\}$

   b. $A = \{2, 4, 6, 8, 10, 12, ...\}, B = \{2, -2, 3, -3, 4, -4, ...\}$

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a bijection. Prove that $g(x) = 3f(x) - 1$ is also a bijection.

3. Consider two sets A and B, and a surjection $f : A \to B$. In this question, we will prove that there always exists a subset S_A of A such that $f : S_A \to B$ is a bijection.

   a. It helps to start with an example to figure out exactly what the statement means. Come up with two sets A and B and a function $f : A \to B$ that is surjective. Try to then find a subset of A and a subset of B such that f is a bijection between the subsets. Is it possible?

   b. Now try to prove the result you found in the first part. Remember, in your proof you need to argue about general sets A and B and a general function f. Examples are only for us to develop an intuition about the problem; they are not proofs.

4. Rewrite the following statements using propositional logic. You do not need to prove them to be true/false.

   a. There exists an integer solution to the equation $x^2 + 5x + 5 = 0$.

   b. There are no three positive integers a, b, c that satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than 2.

   c. If r is a real number such that $|r| < 1$, the sum of the series $1 + r + r^2 + r^3 + \cdots$ is equal to $\dfrac{1}{1-r}$.

5. Rewrite the following statements in English. Again, you do not need to prove them to be true/false. (In part c, let $\mathbb{P}$ represent the set of all primes.)

   a. $\forall n \in \mathbb{N},\ \exists p \in \mathbb{R}_{>0} \mid (n + p = 0)$

b. $\exists x \in \mathbb{R} : x \notin \mathbb{Q}$

c. $\forall n \in \mathbb{N}, n > 1, n \notin \mathbb{P}, \exists\, p \in \mathbb{P} : \frac{n}{p} \in \mathbb{N}$

6. Determine the contrapositive and converse of each of the following statements.

   a. If it is cold outside, then I wear a sweater.

   b. If there is a fire or there is an earthquake, the alarm goes off.

   c. If two distinct natural numbers share a common factor, then at least one of them is composite.

   d. If the contrapositive of a statement is true, then its converse is also true.

7. Using truth tables, prove that the following equivalences hold:

   a. $\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$

   b. $P \implies Q \equiv (\neg P \vee Q)$

   c. $P \implies (Q \wedge \neg R) \equiv \neg P \vee \neg(\neg Q \vee R)$

8. Suppose that $P(x)$ and $Q(x, y)$ are predicates. Find the negation of the following statements:

   a. $\exists x P(x)$

   b. $\forall x P(x)$

   c. $(\forall x)(\exists y) Q(x, y)$

   d. $(\exists x)(\forall y) Q(x, y)$

   Hint: replace $P(x)$ with a real propositional statement, i.e. "x is even."