Problem Set 2: Number Sets, Propositional Logic

CS 198-087: Introduction to Mathematical Thinking
UC Berkeley EECS
Fall 2018

This homework is due on Monday, September 17th, at 6:30PM, on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX.

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1. Determine a bijection $f : A \to B$ between each pair of sets, and prove that f is a bijection (i.e. show that it is both an injection and surjection).

   a. $A = \{1, 2, 3, 4, 5, 6, ...\}, B = \{4, 7, 10, 13, 16, 19, ...\}$

   b. $A = \{2, 4, 6, 8, 10, 12, ...\}, B = \{2, -2, 3, -3, 4, -4, ...\}$

   Solution:

   a. $f(x) = 3x + 1$. This is a linear function, and linear functions are bijections. To see this, draw a picture.

   b. $f(x) = \begin{cases} \frac{1}{4}x + \frac{3}{2} & x \neq 4k, k \in \mathbb{Z}^+ \\ -\frac{1}{4}x - 1 & x = 4k, k \in \mathbb{Z}^+ \end{cases}$
   This is very similar to the bijection between whole numbers and integers we saw in lecture. No two inputs map to the same output, and each integer such that $|n| >= 2$ will be seen as an output at some point.

   In both cases, we can use the method of finding the equation of a line between two points to find the linear functions.

2. Write the equivalent of the following statements using propositional logic.

   a. There exists an integer solution to the equation $x^2 + 2x + 1 = 0$.

   b. There are no three positive integers a, b, c that satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than 2.

   c. $\sqrt{2}$ is a rational number.

   d. For $|r| < 1, r \in \mathbb{R}$, the sum of the series $1 + r + r^2 + r^3 + \cdots$ is equal to $\dfrac{1}{1-r}$.

Solution:

a. $(\exists x \in \mathbb{Z})(x^2 + 2x + 1 = 0)$.

b. $\neg(\exists a, b, c, n \in \mathbb{Z}^+)(n > 2 \wedge a^n + b^n = c^n)$.

c. $(\exists a, b \in \mathbb{Z})(\sqrt{2} = \dfrac{a}{b})$.

d. $(\forall r \in \mathbb{R} s.t. |r| < 1)(\sum_{n=0}^{\infty} r^n = \dfrac{1}{1-r})$.

3. Determine the contrapositive and converse of each of the following statements.

   a. If it rains tomorrow, I will bring an umbrella.

   b. If the clock is between 12PM and 2PM and I am hungry, then it is lunch time.

   c. If your final grade in this course is at least 70%, you will pass.

   d. If two sets A, B are disjoint, then the cardinality of their intersection is 0.

Solution:

a. *Contrapositive:* I will not bring an umbrella if it does not rain tomorrow.
Converse: I will bring an umbrella if it rains tomorrow.

b. *Contrapositive:* It is not lunchtime if the clock is not between 12PM and 2PM or I am not hungry. (notice how the second "and" was turned into an "or")
Converse: It is lunchtime if the clock is between 12PM and 2PM and I am hungry.

c. *Contrapositive:* You will not pass if your final grade in this course is not at least 70%.
Converse: You will pass if your final grade in this course is at least 70%.

d. *Contrapositive:* The cardinality of the intersection of two sets A, B is not 0 if they are not disjoint.
Converse: The cardinality of the intersection of two sets A, B is 0 if they are disjoint.

4. Using truth tables, prove that the following equivalences hold:

   a. $\neg(P \wedge Q) \equiv (\neg P \vee \neg Q)$

   b. $\neg(P \vee Q) \equiv (\neg P \wedge \neg Q)$

   c. $P \implies Q \equiv \neg P \vee Q$

Solution:

1.

P	Q	$\neg(P \wedge Q)$	$(\neg P \vee \neg Q)$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

2.

P	Q	$\neg(P \vee Q)$	$(\neg P \wedge \neg Q)$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

3.

P	Q	$P \implies Q$	$(\neg P \vee Q)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

5. Suppose that $P(x)$ and $Q(x, y)$ are propositional statements. Find the negation of the following statements:

   a. $\exists x P(x)$

   b. $\forall x P(x)$

   c. $(\forall x)(\exists y)Q(x, y)$

   d. $(\exists x)(\forall y)Q(x, y)$

   Hint: replace $P(x)$ with a real propositional statement, i.e. "x is even."

Solution:

a. $\forall x \neg P(x)$

   For example, suppose we're dealing with all students in this course, and $P(x)$ is the proposition "student x is a sophomore". This equivalence says that the statements "there does not exist a student that is a sophomore" and "all students are not sophomores" are equivalent statements.

b. $\exists x \neg P(x)$

   Continuing with the above example, this equivalence states that "if not all students in this course are sophomores" and "there exists a student in this course who is not a sophomore” are equivalent statements.

c. $(\exists x)(\forall y)\neg Q(x,y)$

Suppose $Q(x,y)$ is the multivariate proposition ”$y = x^2$” (for example, $Q(3,4)$ is the proposition that $4 = 3^2$, which is false), and suppose we're dealing with the universe of the real numbers. The negation of our original statement, $\neg((\forall x)(\exists y)\neg Q(x,y))$, states that ”it is not the case that every x has some y such that $y = x^2$”, i.e. ”it is not the case that every real number x has as square.” The equivalent of this statement, $(\exists x)(\forall y)\neg Q(x,y)$, is ”there exists some x, such that for all y, $y \neq x^2$”, i.e. ”there exists some real number x that does not have a square.” These two statements are saying the same thing: if not all real numbers have squares, there must exist some real number without a square.

d. $(\forall x)(\exists y)\neg Q(x,y)$

Let's use the $Q(x,y)$ defined above. The negation of our original statement,

$\neg((\exists x)(\forall y)Q(x,y))$, states that ”it is not the case that there exists some x such that for all y, $y = x^2$”, i.e. ”there is no x that satisfies $y = x^2$ for all y.” The equivalent of this statement, $(\forall x)(\exists y)\neg Q(x,y)$, says that ”every x has some y such that $y \neq x^2$.” Again, these two statements are saying the same thing: if there is no x that satisfies $y = x^2$ for every single y, then every x has some value of y such that $y \neq x^2$.

These statements are slightly difficult to parse. Make sure you read them carefully!

6. In general, you cannot reverse the order of different existential quantifiers. That is,

$$(\forall x)(\exists y)P(x,y) \not\equiv (\exists y)(\forall x)P(x,y)$$

Give two examples of propositions $P(x,y)$ that illustrate why this does not hold.

Solution: Consider $P(x,y) : y = x^2$ (note here we're implicitly letting our universe $\mathbb{U} = \mathbb{R}$). The first statement says that for all x, there is some y such that $y = x^2$. In other words, it says that every real number x has a square. This statement happens to be true.

The second statement says that there is some magical y that is equal to the square of every single x. That is, this y (let's call it y_0) is such that $y_0 = 1^2, y_0 = \pi^2, y_0 = 100^2, \ldots.$ This statement is definitely very false.

These are both saying very different things!