PROBLEM SET 2: NUMBER SETS, PROPOSITIONAL LOGIC

CS 198-087: INTRODUCTION TO MATHEMATICAL THINKING
UC BERKELEY EECS
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This homework is due on Friday, February 15, 11:59 PM, on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX.

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1. Determine a bijection $f : A \to B$ between each pair of sets, and prove that f is a bijection. *(While this can be done by showing that such a mapping is invertible, and therefore bijective, the standard way we'll proceed in this class is by showing a function is both injective and surjective.)*

   a. $A = \{1, 2, 3, 4, 5, 6, ...\}, B = \{4, 7, 10, 13, 16, 19, ...\}$

   b. $A = \{2, 4, 6, 8, 10, 12, ...\}, B = \{2, -2, 3, -3, 4, -4, ...\}$

Solution:

To start, we recognize the fact that any function of the form is both injective and surjective. This is given in the textbook in Section 1.2.

a. $f(x) = 3x + 1$.

b. $f(x) = \begin{cases} \frac{1}{4}x + \frac{3}{2} & x \neq 4k, k \in \mathbb{Z}^+ \\ -\frac{1}{4}x - 1 & x = 4k, k \in \mathbb{Z}^+ \end{cases}$

This is very similar to the bijection between whole numbers and integers we saw in lecture. No two inputs map to the same output, and each integer such that $|n| >= 2$ will be seen as an output at some point.

In both cases, we can use the method of finding the equation of a line between two points to find the linear functions.

2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a bijection. Prove that $g(x) = 3f(x) - 1$ is also a bijection.

Solution: To show that it is a bijection, we'll show that it is both an injection and surjection.

Injection

We need to show $g(a) = g(b) \implies a = b$. We can start by assuming $g(a) = g(b)$. Then,

$$
\begin{aligned}
g(a) &= g(b) \\
3f(a) - 1 &= 3f(b) - 1 \\
f(a) &= f(b) \quad \textit{(by adding 1 and dividing by 3 on both sides)} \\
\implies a &= b \quad \textit{(since } f \textit{ is injective)}
\end{aligned}
$$

as required.

Surjection

We now need to show that for any $c \in \mathbb{R}$, there is some b such that $g(b) = c$.

$$
\begin{aligned}
g(b) &= c \\
3f(b) - 1 &= c \\
f(b) &= \frac{c+1}{3}
\end{aligned}
$$

Now, since f is a surjection, we know that for any $s \in \mathbb{R}$, there is some number in $\mathbb{R}$ that we can pass into f as an input to get r as an output. In our case, we have that if we pass b into f, the result is $\frac{c+1}{3}$, implying that passing b into g yields c as an output. Since c is any arbitrary real number, this tells us that g is indeed surjective.

Since we've shown g to be both injective and surjective, we've proved that g is a bijection.

3. Consider two sets A and B, and a surjection $f : A \to B$. In this question, we will prove that there always exists a subset S_A of A such that $f : S_A \to B$ is a bijection.

   a. It helps to start with an example to figure out exactly what the statement means. Come up with two sets A and B and a function $f : A \to B$ that is surjective. Try to then find a subset of A and a subset of B such that f is a bijection between the subsets. Is it possible?

   b. Now try to prove the result you found in the first part. Remember, in your proof you need to argue about general sets A and B and a general function f. Examples are only for us to develop an intuition about the problem; they are not proofs.

Solution:

a. Consider the sets $A = \{0, \frac{\pi}{2}, \pi\}$ and $B = \{0, 1\}$, and the function $f(x) = \sin(x)$. f is a surjection from A to B but not a bijection as both $\sin(0) = \sin(\pi) = 0$.
Now consider the subset of A, $S_A = \{0, \frac{\pi}{2}\}$. Clearly, $f : S_A \to B$ is a bijection, since $\sin(0) = 0$ and $\sin(\frac{\pi}{2}) = 1$.

b. First, we'll give an example of the general principle, then provide a formal proof.

Suppose $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 3, 4\}$. Suppose f is defined as $f(1) = f(2) = 2$, $f(3) = f(4) = 3$, and $f(5) = 4$. Then we can make a new set S_A containing exactly one of 1 and 2, 3 and 4, and 5, since 1 and 2 map to the same element of B, as do 3 and 4. We must include 5, otherwise $f : S_A \to B$ will not be surjective (as we will not be able to hit the value 4 in B). One possible S_A is $\{1, 3, 5\}$. Then, $f : S_A \to B$ becomes $f(1) = 2$, $f(3) = 3$, and $f(5) = 4$. Clearly, f is both an injection and a surjection from S_A to B, and $S_A \subseteq A$.

Now we formalize the proof.

Consider two sets A and B and a surjection $f : A \to B$ such that f is not a bijection. Since f is a surjection but not a bijection, it must not be an injection.

This means, for every element $b \in B$, there is at least one element $a \in A$ such that $f(a) = b$. To formalize things further, suppose for any $b_i \in B$, the elements in A that map to b_i are given by $a_{i_1}, a_{i_2}, ...$

This means there are unique elements $a_1, a_2, \ldots, a_n \in A : f(a_1) = f(a_2) = \ldots = f(a_n) = b_1 \in B$ (i.e., f maps multiple inputs to the same output). For each $b_i \in B$, we can choose any one of $a_1, \ldots, a_n$ and put it in S_A (note that $n = 1$ is a possibility). Observe that S_A is a subset of A since all of its elements are also elements of A, and also observe that there exist no $x, y \in S_A : f(x) = f(y)$ since we chose the elements of S_A in such a way that they only map to unique $b_i \in B$. Thus, $f : S_A \to B$ is injective. Since we did this for all elements in B, $f : S_A \to B$ is a surjection. Thus, $f : S_A \to B$ is a bijection, since it is both surjective and injective.

4. Rewrite the following statements using propositional logic. You do not need to prove them to be true/false.

   a. There exists an integer solution to the equation $x^2 + 5x + 5 = 0$.

   b. There are no three positive integers a, b, c that satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than 2.

   c. If r is a real number such that $|r| < 1$, the sum of the series $1 + r + r^2 + r^3 + \cdots$ is equal to $\dfrac{1}{1-r}$.

Solution:

a. $(\exists x \in \mathbb{Z})(x^2 + 5x + 5 = 0)$

b. $\neg(\exists a, b, c, n \in \mathbb{Z}^+)(n > 2 \land a^n + b^n = c^n)$

c. $(\forall r \in \mathbb{R} : |r| < 1)(\sum_{n=0}^{\infty} r^n = \dfrac{1}{1-r})$

If you were curious, the first statement is false, the second is true, and the third is also true.

5. Rewrite the following statements in English. Again, you do not need to prove them to be true/false. (In part c, let $\mathbb{P}$ represent the set of all primes.)

   a. $\forall n \in \mathbb{N},\ \exists p \in \mathbb{R}_{>0} \mid (n + p = 0)$

   b. $\exists x \in \mathbb{R} : x \notin \mathbb{Q}$

   c. $\forall n \in \mathbb{N}, n > 1, n \notin \mathbb{P}, \exists\, p \in \mathbb{P} : \frac{n}{p} \in \mathbb{N}$

> **Solution:**
>
> a. For all natural numbers n, there exists a positive real number p such that their sum is zero. numbers cannot possibly be zero.
>
> b. There exists a real number x such that x is not a rational number.
>
> c. For all non-prime natural numbers n greater than 1, there exists some other prime number p such that $\frac{n}{p}$ is also a natural number.
>
> *If you were curious, the first statement is false, the second is true, and the third is also true.*

6. Determine the contrapositive and converse of each of the following statements.

   a. If it is cold outside, then I wear a sweater.

   b. If there is a fire or there is an earthquake, the alarm goes off.

   c. If two distinct natural numbers share a common factor, then at least one of them is composite.

   d. If the contrapositive of a statement is true, then its converse is also true.

> **Solution:**
>
> a. *Contrapositive:* If I don't wear a sweater, then it is not cold outside.
> *Converse:* If I wear a sweater, then it is cold outside.
>
> b. *Contrapositive:* If the alarm does not go off, then there is no fire and there is no earthquake. (Note how the *or* turned into an *and* after negation.)
> *Converse:* If the alarm goes off, then there is a fire or an earthquake.
>
> c. *Contrapositive:* If neither of two distinct natural numbers is composite, then they do not share any common factors. (Note how *at least one* turned into *neither of*.)
> *Converse:* If at least one of two distinct natural numbers is composite, then they share a common factor.
>
> d. *Contrapositive:* If the converse of a statement is not true, then its contrapositive is also not true. (This, and the original statement, are not true.) *Converse:* If the converse of a statement is true, then its contrapositive is also true.

7. Using truth tables, prove that the following equivalences hold:

a. $\neg(P \land Q) \equiv (\neg P \lor \neg Q)$

b. $P \implies Q \equiv (\neg P \lor Q)$

c. $P \implies (Q \land \neg R) \equiv \neg P \lor \neg(\neg Q \lor R)$

Solution:

1.

P	Q	$\neg(P \land Q)$	$(\neg P \lor \neg Q)$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	T	T

2.

P	Q	$P \implies Q$	$(\neg P \lor Q)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

3.

P	Q	R	$P \implies (Q \land \neg R)$	$\neg P \lor \neg(\neg Q \lor R)$
T	T	T	F	F
T	T	F	T	T
T	F	T	F	F
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

8. Suppose that $P(x)$ and $Q(x, y)$ are predicates. Find the negation of the following statements:

a. $\exists x P(x)$

b. $\forall x P(x)$

c. $(\forall x)(\exists y)Q(x, y)$

d. $(\exists x)(\forall y)Q(x, y)$

Hint: replace $P(x)$ with a real propositional statement, i.e. "x is even."

Solution:

a. $\forall x \neg P(x)$

For example, suppose we're dealing with all students in this course, and $P(x)$ is the proposition "student x is a sophomore". This equivalence says that the statements "there does not exist a student that is a sophomore" and "all students are not sophomores" are equivalent statements.

b. $\exists x \neg P(x)$

Continuing with the above example, this equivalence states that "if not all students in this course are sophomores" and "there exists a student in this course who is not a sophomore" are equivalent statements.

c. $(\exists x)(\forall y)\neg Q(x, y)$

Suppose $Q(x, y)$ is the multivariate proposition "$y = x^2$" (for example, $Q(3, 4)$ is the proposition that $4 = 3^2$, which is false), and suppose we're dealing with the universe of the real numbers. The negation of our original statement, $\neg((\forall x)(\exists y)\neg Q(x, y))$, states that "it is not the case that every x has some y such that $y = x^2$", i.e. "it is not the case that every real number x has as square." The equivalent of this statement, $(\exists x)(\forall y)\neg Q(x, y)$, is "there exists some x, such that for all y, $y \neq x^2$", i.e. "there exists some real number x that does not have a square." These two statements are saying the same thing: if not all real numbers have squares, there must exist some real number without a square.

d. $(\forall x)(\exists y)\neg Q(x, y)$

Let's use the $Q(x, y)$ defined above. The negation of our original statement,

$\neg((\exists x)(\forall y)Q(x, y))$, states that "it is not the case that there exists some x such that for all y, $y = x^2$", i.e. "there is no x that satisfies $y = x^2$ for all y." The equivalent of this statement, $(\forall x)(\exists y)\neg Q(x, y)$, says that "every x has some y such that $y \neq x^2$." Again, these two statements are saying the same thing: if there is no x that satisfies $y = x^2$ for every single y, then every x has some value of y such that $y \neq x^2$.

These statements are slightly difficult to parse. Make sure you read them carefully!