PROBLEM SET 3: PROOF TECHNIQUES

CS 198-087: INTRODUCTION TO MATHEMATICAL THINKING
UC BERKELEY EECS
FALL 2018

This homework is due on Monday, September 24th, at 6:30PM, on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX.

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1. Prove that if $x^2 - 6x + 5$ is odd, then x is even, using:
   a. Proof by Contraposition
   b. Direct Proof
2. Prove that if $P \implies Q$ and $R \implies \neg Q$, then $P \implies \neg R$, using:
   a. Proof by Contradiction
   b. Proof by Contraposition *(Hint: You may need to use the fact that $A \implies B \equiv \neg A \vee B$.)*
3. In the previous problem set, we had you prove De Morgan's Laws for conjunctions and disjunctions.

$$\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$$
$$\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$$

   a. *(Optional)* Using truth tables, prove both of the statements above.
   b. In terms of the symbols $\vee, \wedge, \neg$, re-write the *negation* of $P \implies Q$. *(Hint: We did this in lecture, but try and derive it on your own.)*
   c. Now, let's consider the statement "if $a \cdot b$ is even, then a is even or b is even." First, rewrite this statement using mathematical notation. *(Hint: you will need the implication symbol.)*
   d. Find the contrapositive of the statement in (c), and write it using mathematical notation.
   e. Prove the statement in (c) by contraposition.
   f. Prove the statement in (c) using a direct proof. *(Hint: You may need to consider multiple cases.)*
   g. Why do you think we included this problem?

4. In this problem, we'll look at how to prove statements of the form "P if and only if Q." Recall, the statement $P \iff Q$ is equivalent to $(P \implies Q) \wedge (Q \implies P)$. To show that P is true if and only if Q is true, we need to prove both directions of the implication, i.e. that P implies Q and that Q implies P. We can use any proof technique we like for each of the two sub-proofs.

   For example: Suppose we want to prove that $|A \cup B| = |A| + |B|$ if and only if A and B are disjoint. We need to prove two directions: if $|A \cup B| = |A| + |B|$, then A and B are disjoint (1), and if A and B are disjoint, then $|A \cup B| = |A| + |B|$ (2).

   a. Complete the proof above.

   b. We say (a, b, c) is a *Pythagorean triplet* if $a^2 + b^2 = c^2$. Prove that (a, b, c) is a Pythagorean triplet if and only if at least one of a, b, c are even.

5. When we are presented with the task of proving the uniqueness of two elements (where elements could be numbers, sets, vectors, etc.) we usually use a proof by contradiction. We assume that there exist two different instances of such an element, and show that they must be the same.

   For example, suppose we want to prove that additive inverses are unique. (The inverse of some real number x for the operation of addition is $-x$, as $x + (-x) = 0$.) We could start by assuming that there exist two inverses, a and b, such that $a \neq b$. This means that $x + a = 0$ and $x + b = 0$. Then:

$$\begin{aligned} a &= a \\ &= a + (x + b) \\ &= (a + x) + b \\ &= 0 + b \\ &= b \end{aligned}$$

   We started by assuming $a \neq b$, but showed that $a = b$, disproving the notion that additive inverses are unique.

   Sometimes, the notation $\exists!$ is used to mean "there exists a unique" / "there exists exactly one." For example, $\exists!\, x \in \mathbb{R} \mid x^2 - 4x + 4 = 0$ translates to "there exists a unique real solution to $x^2 - 4x + 4 = 0$."

   a. Prove, using a technique similar to that above, that there only exists a single positive integer solution to $a^2 + 4^2 = 5^2$.

   b. Prove that $(\forall x \in \mathbb{R}, x \neq 0)(\exists!\, y \in \mathbb{R})(x \cdot y = 1)$.

6. Over the summer, Billy spent all day thinking about math (same!) and trying to come up with the next big proof. After a few months, he presents this proof to you to proofread. Verify that his proof is correct, or explain the error.

   Theorem: If n is an integer and $2n + 2$ is even, then n is odd.

Proof: Proceed by contraposition. Assume that n is odd. We will now prove that $2n + 2$ is even. Clearly, $2n$ must be an even number, since it is divisible by 2. Furthermore, 2 is an even number, so $2n + 2$ must be even. This concludes the proof.

7. The following problems involve parity arguments that is, reasoning about whether some combinations of integers are even or odd.

   a. Prove that if a, b and c are odd integers, then there are no integer solutions to $ax^2+bx+c = 0$. *(Hint 1: You should consider two different cases for x. Hint 2: Negative integers can still be even or odd. Hint 3: Is it possible for the sum of a few integers to be a non-integer?)*

   b. Prove that there are no integer solutions to $a^2 - 4b = 2$. *(Hint: How can we formulate this as a Proof by Contraposition?)*

8. Prove that $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}} = 2$. *(This is a challenging problem. Try not to search for it online. Hint: Define some variable x recursively.)*