Problem Set 3: Proof Techniques

CS 198-087: Introduction to Mathematical Thinking
UC Berkeley EECS
Spring 2019

This homework is due on Friday, February 22, at 11:59 PM on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX.

---

1. Prove that the product of two odd numbers is odd.
2. a. Prove that if x^2 is even, then x is even.

   b. Prove that if x is even, then x^2 is even.

   c. Using the above two proofs, conclude that x^2 is even if and only if x is even.
3. Prove that if $x^2 - 8x + 12$ is even, then x is even, using:

   a. Proof by Contraposition

   b. Direct Proof
4. Let x and y be positive integers. Prove that if $x \times y < 25$, then $x < 5$ or $y < 5$.
5. Given $a, b, x, y \in \mathbb{N}$ such that $A = a + \frac{1}{x}$, $B = b + \frac{1}{y}$, y divides a, and x divides b, prove that the product of A and B is an integer if and only if $x = y = 1$.
6. Prove that there are no integer solutions to $a^2 - 4b = 2$.
7. Prove that there are no $x, y \in \mathbb{N}$ such that $x^2 - y^2 = 1$.
8. Prove that if $P \implies Q$ and $R \implies \neg Q$, then $P \implies \neg R$, using:

   a. Proof by Contradiction

   b. Proof by Contraposition

   (Hint: It may help to think of the statement "$P \implies Q$ and $R \implies \neg Q$" as A, and "$P \implies \neg R$" as B.)
9. Over the summer, Billy decided he should practice what he learnt from the IMT DeCal in order to be fresh and ready for CS70 during the Fall. He vaguely remembers something on implications, and decides to write a proof on it. Verify that his proof is correct, or explain the error.

   Theorem: If $A \implies B$ is True, then A is False.

Proof: Proceed by contraposition. Assume that A is False. Looking at the truth table for $A \implies B$:

A	B	$A \implies B$
T	T	T
T	F	F
F	T	T
F	F	T

We can see that when A is False, the implication is True. This concludes the proof.

10. *(Optional)* Prove that $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}} = 2$. *(This is a challenging problem. Try not to search for it online. Hint: Define some variable x recursively.)*