# PROBLEM SET 3: PROOF TECHNIQUES

**CS 198-087: INTRODUCTION TO MATHEMATICAL THINKING**
UC BERKELEY EECS
SPRING 2019

This homework is due on Friday, February 22, at 11:59 PM on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX.

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1. Prove that the product of two odd numbers is odd.

> **Solution:** We can write any odd number in the form $2k+1$, where $k \in \mathbb{Z}^+$. Let $a = 2n+1$ and $b = 2m+1$. Then, $a \cdot b = (2n+1)(2m+1) = 4nm + 2(n+m) + 1 = 2(2nm+n+m)+1$. We've shown that $a \cdot b$ can be written as $2 \cdot$(some integer)$+1$, thus showing that the product of two odd numbers is odd.

2. a. Prove that if $x^2$ is even, then $x$ is even.
   b. Prove that if $x$ is even, then $x^2$ is even.
   c. Using the above two proofs, conclude that $x^2$ is even if and only if $x$ is even.

> **Solution:**
>
> a. Let's proceed by contraposition. The contrapositive of this statement is "if $x$ is odd, then $x^2$ is odd." If $x$ is odd, we can write $x = 2k+1$, where $k \in \mathbb{Z}$. Now, let's consider $x^2$:
>
> $$\begin{aligned} x^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$
>
> Since we can write $x^2 = 2 \cdot (\text{some integer}) + 1$, we can conclude that if $x$ is odd, then $x^2 isodd$. But, this was the contraposition of our original statement, which therefore also holds.
>
> b. We can do this directly. Since $x$ is even, we can say $x = 2k$. Then, $x^2 = (2k)^2 = 4k^2 = 2(2k^2)$. We've shown $x^2 = 2 \cdot (\text{some integer})$, therefore the statement holds.

c. The implication "$x^2$ is even if and only if $x$ is even" breaks down into two separate implications: "if $x$ is even, then $x^2$ is even" (which we proved in part b) and "if $x^2$ is even, then $x$ is even" (which we proved in part a).

We've shown both implications to be true. Therefore, we can say $x^2$ is even if and only if $x$ is even.

3. Prove that if $x^2 - 8x + 12$ is even, then $x$ is even, using:
   a. Proof by Contraposition
   b. Direct Proof

**Solution:**

a. The contrapositive of this statement is, "if $x$ is odd, then $x^2 - 8x + 12$" is odd. If $x$ is odd, we can write $x = 2k + 1$ for $k \in \mathbb{Z}$. Then:

$$\begin{aligned} x^2 - 6x + 5 &= (2k+1)^2 - 8(2k+1) + 12 \\ &= (4k^2 + 4k + 1) - (16k + 8) + 12 \\ &= 2(2k^2 - 12k + 4) + 1 \end{aligned}$$

We've shown that when $x$ is odd, $x^2 - 8x + 12$ can be written as $2 \cdot$ (some integer) + 1, therefore by contraposition, the original statement holds.

b. There are several ways to do this directly.

- We can factor $x^2 - 8x + 12$ into $(x-6)(x-2)$. If this product is odd, then each of $x - 6$ and $x - 2$ has to be odd (the only way for a product of two integers to be odd is if both integers are odd). But if $x - 6$ is odd, then $x$ has to be even (same with $x - 2$), as adding one to any odd integer makes it even.
- We can break this into two cases: when $x$ is even and when $x$ is odd. For each, we can evaluate $x^2 - 8x + 12$; this will show that when $x$ is even, $x^2 - 8x + 12$ is even, and when $x$ is odd, $x^2 - 8x + 12$ is odd. This also proves the original statement.

4. Let $x$ and $y$ be positive integers. Prove that if $x \times y < 25$, then $x < 5$ or $y < 5$.

**Solution:** *Credit: http://practice.eecs70.org*

**Proof by Contraposition.**

*Goal:* If $x \times y < 25$, then $x < 5$ or $y < 5$.

*Contrapositive:* If $x \geq 5$ and $y \geq 5$, then $x \times y \geq 25$.

*Approach:* Assume that $x \geq 5$ and $y \geq 5$. Multiple both equations together:

$x \times y \geq 5 \times 5 \equiv x \times y \geq 25$

*Conclusion:* If $x \geq 5$ and $y \geq 5$, then $x \times y \geq 25$. Therefore, the equivalent implication – if $x \times y \leq 25$, then $x < 5$ or $y < 5$ – is also true.

5. Given $a, b, x, y \in \mathbb{N}$ such that $A = a + \frac{1}{x}$, $B = b + \frac{1}{y}$, $y$ divides $a$, and $x$ divides $b$, prove that the product of $A$ and $B$ is an integer if and only if $x = y = 1$.

**Solution:** To prove an "if and only if" statement, there are two statements we need to prove: Given the initial conditions, ...

1. if $x = y = 1$, then the product of $A$ and $B$ is an integer.
2. if the product of $A$ and $B$ is an integer, then $x = y = 1$.

Let's prove the first statement. If $x = y = 1$, then $A = a + 1$ and $B = b + 1$ by subsitution. Thus, the product of $A$ and $B$ would be: $AB = (a + 1)(b + 1) = ab + a + b$. Since all of the components of this product are integers, we have proven that the product of $A$ and $B$ is also an integer.

Now, let's prove the second statement. Firstly, let's expand the product of $A$ and $B$: $AB = (a + \frac{1}{x})(b + \frac{1}{y}) = ab + \frac{b}{x} + \frac{a}{y} + \frac{1}{xy}$. Using the initial condition that $y$ divides $a$ and that $x$ divides $b$, we know that $\frac{b}{x}$ and $\frac{a}{y}$ are both integers. However, we require that $\frac{1}{xy}$ to also be an integer for the product of $A$ and $B$ to be an integer. This can only be satisifed if $x = y = 1$.

6. Prove that there are no integer solutions to $a^2 - 4b = 2$.

**Solution:** We will proceed by contradiction. Let's assume there are integral solutions for $a, b$.

We can rearrange to have $a^2 = 2 + 4b = 2(1 + 2b)$. This means that $a^2$ is even, and thus $a$ is even, meaning we can say $a = 2k$.

Then,

$$\begin{aligned} (2k)^2 &= 2(1 + 2b) \\ 4k^2 &= 2(1 + 2b) \\ 2k^2 &= 1 + 2b \end{aligned}$$

Here, we have a contradiction, as $2k^2$ is even, but $1 + 2b$ must be odd. Therefore, there cannot be any integer solutions for $a, b$.

7. Prove that there are no $x, y \in \mathbb{N}$ such that $x^2 - y^2 = 1$.

**Solution:** Let's proceed by contradiction. Assume that there exist some natural numbers $x, y$ that satifies the given equation.

We can factor the left-hand side using the Difference of Squares, $x^2 - y^2 = (x - y)(x + y)$. Note that since $x \geq 1$ and $y \geq 1$, $x + y \geq 2 \in \mathbb{N}$.

Then, dividing both sides of the equation by $x + y$ gives us

$$x - y = \frac{1}{x + y}$$

The left hand side, $x - y$, is an integer, but not necessarily a natural number (for example, $2, 3 \in \mathbb{N}$ but $2 - 3 \notin \mathbb{N}$).

The right hand side, though, is a rational number, since $x + y \geq 2$. We can prove this rigorously: Starting with $x + y \geq 2$, we can divide both sides by $x + y$ and 2 to give us $\frac{1}{2} \geq \frac{1}{x+y}$. Furthermore, since 1 and $x + y$ are both positive, their ratio is positive. Thus, we have $\frac{1}{2} \geq \frac{1}{x+y} > 0$. There are no integers greater than 0 and less than $\frac{1}{2}$, thus $\frac{1}{x+y}$ is rational.

This is a contradiction! We've stated that an integer is equal to a non-integer rational number. This is impossible. Therefore, by contradiction, there are no natural solutions to $x^2 - y^2 = 1$.

8. Prove that if $P \implies Q$ and $R \implies \neg Q$, then $P \implies \neg R$, using:

   a. Proof by Contradiction

   b. Proof by Contraposition *(Hint: You may need to use the fact that $A \implies B \equiv \neg A \vee B$.)*

**Solution:** It may help to think of the statement "$P \implies Q$ and $R \implies \neg Q$" as $A$, and "$P \implies \neg R$" as $B$.

a. The negation of the statement $A \implies B$ is $A \wedge \neg B$ (this was seen in lecture, but you should prove it to yourself). So, the negation of our statement is that $P \implies Q$ and $R \implies \neg Q$, but $\neg(P \implies \neg R)$. We can rewrite $\neg(P \implies \neg R)$ as $\neg(\neg P \vee \neg R) \equiv P \wedge R$. In other words, we are now assuming that all of the following are true:

   - $P \implies Q$
   - $R \implies \neg Q$
   - $P \wedge R$

From the statement $P \wedge R$, we know that $P$ and $R$ both must be true. If $P$ is true, then $Q$ must be true in order for $P \implies Q$ to be true. However, this means the statement $R \implies \neg Q$ is false, as a true value ($R$) does not imply a false value ($\neg Q$). This is a contradiction.

This means that the negation of our original statement cannot be true, and therefore our original statement was true.

b. The contraposition of our original statement is

$$\begin{aligned}
\neg(P \implies \neg R) &\implies \neg((P \implies Q) \wedge (R \implies \neg Q)) \\
(P \wedge R) &\implies \neg((\neg P \vee Q) \wedge (\neg R \vee \neg Q)) \\
(P \wedge R) &\implies ((P \wedge \neg Q) \vee (R \wedge Q))
\end{aligned}$$

We now need to show that this implication has a true value. This implication is true when:

- $P \wedge R$ is true **and** $((P \wedge \neg Q) \vee (R \wedge Q))$ is true
- $P \wedge R$ is false

There isn't a whole lot to prove when $P \wedge R$ is false (as "false" $\implies$ "true" and "false" $\implies$ "false" are both true).

Suppose $P \wedge R$ is true, meaning $P$ and $R$ are both true. We have to consider both cases of $Q$: $Q$ could either be true or false.

| $P$ | $R$ | $Q$ | $(P \wedge \neg Q) \vee (R \wedge Q)$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ |

You should verify the above two cases yourself. In short, when $Q$ is true, $P \wedge \neg Q$ is false and $R \wedge Q$ is true. When $Q$ is false, $P \wedge \neg Q$ is true and $R \wedge Q$ is false. Since these two conditions are joined by a disjunction, only one needs to be true for the entire expression to be true. Therefore, by contraposition, the original statement holds true.

Which of these methods do you think was more straightforward?

9. Over the summer, Billy decided he should practice what he learnt from the IMT DeCal in order to be fresh and ready for CS70 during the Fall. He vaguely remembers something on implications, and decides to write a proof on it. Verify that his proof is correct, or explain the error.

*Theorem:* If $A \implies B$ is True, then $A$ is False.

**Proof:** Proceed by contraposition. Assume that $A$ is False. Looking at the truth table for $A \implies B$:

| $A$ | $B$ | $A \implies B$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ |

We can see that when $A$ is False, the implication is True. This concludes the proof.

**Solution:** The original statement is false. Billy attempted to prove this statement by contraposition, but instead of taking the contrapositive, he took the converse. In general, an implication $P \to Q$ is not logically equivalent with its contrapositive, $Q \to P$.

10. *(Optional)* Prove that $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}} = 2$. *(This is a challenging problem. Try not to search for it online. Hint: Define some variable $x$ recursively.)*

**Solution:** Let $x = \sqrt{2}^x$. Then:

$$\begin{aligned} \ln x &= x \ln 2^{\frac{1}{2}} \\ \frac{1}{x} \ln x &= \frac{1}{2} \ln 2 \end{aligned}$$

It now becomes clear that $x = 2$ satisfies the above relationship. However, it isn't necessarily the only solution. Suppose that $g(x) = \frac{\ln x}{x}$.

By differentiating $g(x)$ it becomes clear that $g(x)$ has a local maximum at $x = e$ and begins decreasing after, implying that there are two solutions potential values of $x$ that could satisfy this equation. It turns out that $x = 4$ is also a solution to $g(x)$.

How do we know that $x = 4$ is not the correct answer?

Suppose we consider some recursive sequence defined by $a_0 = \sqrt{2}$, $a_n = \sqrt{2}^{a_{n-1}}$, it can be shown by induction that $a_n \leq 2$. Do this as an exercise!