PROBLEM SET 4: PROOF TECHNIQUES, MATHEMATICAL INDUCTION

CS 198-087: INTRODUCTION TO MATHEMATICAL THINKING
UC BERKELEY EECS
FALL 2018

This homework is due on Monday, October 1st, at 6:30PM, on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX.

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1. a. Prove, using induction, that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

   b. Prove, using induction, that $\left(\sum_{i=1}^{n} i\right)^2 = \sum_{i=1}^{n} i^3$. *(Hint: In lecture, we showed that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.)*

2. Recall the power rule for derivatives:

$$\frac{d}{dx}x^n = nx^{n-1}$$

   Prove this rule for all $n \in \mathbb{N}_0$ using induction. *(Hint: You will need to use the product rule for derivatives.)*

3. De Moivre's theorem gives us a way to exponentiate complex numbers of the form $R(\cos t + i \sin t)$:

$$(R(\cos t + i \sin t))^n = R^n(\cos nt + i \sin nt)$$

   We can prove this using Euler's theorem, which tells us that $\cos t + i \sin t = e^{it}$.

$$(R(\cos t + i \sin t))^n = (Re^{it})^n = R^n e^{int} = R^n(\cos nt + i \sin nt)$$

   Prove De Moivre's theorem, using induction. *(Hint: You will need the trignometric identities $\cos(a+b) = \cos a \cos b - \sin a \sin b$ and $\sin(a+b) = \sin a \cos b + \cos a \sin b$.)*

4. Consider the following inequality:

$$|\bigcup_{i=1}^{n} A_i| \leq \sum_{i=1}^{n} |A_i|$$

In other words, $|A_1 \cup A_2 \cup ... \cup A_n| \leq |A_1| + |A_2| + ... + |A_n|$.

a. Draw a picture and reason about why this is true intuitively.

b. Prove this using induction.

5. Prove that for $n \geq 3$, the sum of the interior angles of a polygon with n vertices is $180(n-2)$. *(Attempt this problem, but don't spend too long on it, and don't worry if you don't get it. Hint: Drawing pictures will help.)*

6. *Note: This problem may seem very long, but most of it is reading. The work you have to do is relatively little.*

   In this problem, we will look at a new way to derive formulas for sums of the form $1^k + 2^k + ... + n^k$ for any $k \in \mathbb{Z}^+$. You already proved these formulas in the first problem.

   Consider the sum

$$(2-1) + (3-2) + (4-3) + (5-4) + ... + (100-99)$$

   We can rewrite it as

$$-1 + (2-2) + (3-3) + (4-4) + ... + (99-99) + 100$$

   Then, all of the terms other than -1 and 100 cancel out, giving us that the value of this sum is $-1 + 100 = 99$. Such a sum, where part of term i cancels out with part of term $i+1$ and $i-1$, is known as a **telescoping sum**. The act of cancelling these terms is called the **Method of Differences**.

   Another example:

$$\begin{aligned}\sum_{k=1}^{100} \big(\cos(k) - \cos(k-1)\big) &= (\cos 1 - \cos 0) + (\cos 2 - \cos 1) + ... + (\cos 100 - \cos 99) \\ &= -\cos 0 + (\cos 1 - \cos 1) + (\cos 2 - \cos 2) + ... + (\cos 99 - \cos 99) + \cos 100 \\ &= \cos 100 - \cos 0\end{aligned}$$

   How can we use this to get formulas for sums of the form $\sum_{i=1}^{n} i^k$?

   Consider $(i+1)^2 = i^2 + 2i + 1$. We can rearrange this to have $(i+1)^2 - i^2 = 2i + 1$. Then, on the left hand side we have something that resembles a telescoping sum. Since both the left hand side and right hand side are equal, if I sum both sides from $i = 1$ to $i = n$, the results should be the same.

$$
\begin{aligned}
\sum_{i=1}^{n}\left((i+1)^2-i^2\right) &= \sum_{i=1}^{n}\left(2i+1\right) \\
(2^2-1^2)+(3^2-2^2)+...+((n+1)^2-n^2) &= 2\sum_{i=1}^{n}+\sum_{i=1}^{n}1 \\
(n+1)^2-1^2 &= 2\sum_{i=1}^{n}+n
\end{aligned}
$$

We used the fact that $\sum 2x = 2\sum x$ and $\sum_{i=1}^{n} 1 = 1+1+...+1 = n$.

Now, we can simply rearrange for $\sum_{i=1}^{n}$:

$$
\begin{aligned}
(n+1)^2-1^2 &= 2\sum_{i=1}^{n}+n \\
2\sum_{i=1}^{n} &= (n+1)^2-n-1 \\
\sum_{i=1}^{n} &= \frac{n^2+2n+1-n-1}{2} \\
&= \frac{n(n+1)}{2}
\end{aligned}
$$

as we've seen before.

Suppose we wanted to use this technique to find an expression for $\sum_{i=1}^{n} i^2$. We would have to start with the expansion of $(i+1)^3 = i^3+3i^2+3i+1$, rearrange to have $(i+1)^3-i^3 = 3i^2+3i+1$, and take the sum on both sides. This will yield

$$
\sum_{i=1}^{n}\left((i+1)^3-i^3\right) = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1
$$

The left hand side will evaluate to be $-1^3 + (n+1)^3$. The right hand side has both $\sum_{i=1}^{n} i^2$, which we are trying to find, and also $\sum_{i=1}^{n} i$, which we already know. To proceed, you will have to substitute the value $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and rearrange to find $\sum_{i=1}^{n} i^2$.

a. Determine the value of $\sum_{i=1}^{n} \frac{1}{n(n+1)}$. *(Hint: How can you rewrite $\frac{1}{n(n+1)}$ as the difference of two terms?)*

b. Derive $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ using a telescoping sum. Most of this has already been done for you!

c. Derive $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$ using a telescoping sum.