Problem Set 4: Proof Techniques, Induction

CS 198-087: Introduction to Mathematical Thinking
UC Berkeley EECS
Spring 2019

This homework is due on Friday, March 1, at 11:59 PM on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX.

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1. Prove that there are no integer solutions to $x^2 - 3 = 4y$. *(Hint: Break x into two cases.)*

2. a. Prove, using induction, that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

   b. Prove, using induction, that $(\sum_{i=1}^{n} i)^2 = \sum_{i=1}^{n} i^3$. *(Hint: In lecture, we showed that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.)*

3. Prove that $5|n^5 - n, \forall n \in \mathbb{N}$, using each of the following techniques:

   a. Proof by Cases *(Hint: Factor $n^5 - n$, and consider 5 possible cases for n)*

   b. Induction *(Hint: Use the fact that $(c+1)^5 = c^5 + 5c^4 + 10c^3 + 10c^2 + 5c + 1$)*

4. Recall the power rule for derivatives:

$$\frac{d}{dx}x^n = nx^{n-1}$$

   Prove this rule for all $n \in \mathbb{N}_0$ using induction. *(Hint: You will need to use the product rule for derivatives.)*

5. Consider the following inequality:

$$|\bigcup_{i=1}^{n} A_i| \leq \sum_{i=1}^{n} |A_i|$$

   In other words, $|A_1 \cup A_2 \cup ... \cup A_n| \leq |A_1| + |A_2| + ... + |A_n|$.

   Prove this using induction $\forall n \in \mathbb{N}$. *(Hint: You may need to use the Principle of Inclusion-Exclusion for two sets.)*

6. The harmonic series $H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ is known to be unbounded as $n \to \infty$. In this problem, we will use induction to prove that the harmonic series is unbounded.

Using induction, prove that $\forall n \in \mathbb{N}, H_{2^n} \geq 1 + \frac{n}{2}$. Why does this prove that the harmonic series is unbounded?

7. Prove that for $n \geq 3$, the sum of the interior angles of a polygon with n vertices is $180(n-2)$. *(Attempt this problem, but don't spend too long on it, and don't worry if you don't get it. Hint: Drawing pictures will help.)*

8. In this problem, f_i will refer to the Fibonnaci sequence. This sequence is defined by $f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}, \forall n \geq 2, n \in \mathbb{N}$.

   a. Prove that $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$.

   b. Prove that $f_n > 2n$, for $n \geq 8$.

   c. Prove that $f_n \leq 2^n$.

   (Hint: You will need to use strong induction for parts b and c.)