PROBLEM SET 4: PROOF TECHNIQUES, INDUCTION

CS 198-087: INTRODUCTION TO MATHEMATICAL THINKING
UC BERKELEY EECS
SPRING 2019

This homework is due on Friday, March 1, at 11:59 PM on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX.

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1. Prove that there are no integer solutions to $x^2 - 3 = 4y$. *(Hint: Break x into two cases.)*

Solution: Let's proceed by contradiction. Let's assume that there exist integer solutions for x and y. Furthermore, let's consider two cases for x: $x = 2k$ and $x = 2k + 1$, i.e. x is even or odd.

Case 1: $x = 2k$

$$\begin{aligned} (2k)^2 - 3 &= 4y \\ 4k^2 - 3 &= 4y \\ 2(2k^2 - 2y) &= 3 \end{aligned}$$

This is contradictory, as on the left hand side we have an even number ($2 \cdot$ (some integer)), and on the right hand side we have 3, an odd number. This is not possible.

Case 2: $x = 2k + 1$

$$\begin{aligned} (2k+1)^2 - 3 &= 4y \\ 4k^2 + 4k + 1 - 3 &= 4y \\ 4k^2 + 4k - 4y &= 2 \\ 2k^2 + 2k - 2y &= 1 \\ 2(k^2 + k - y) &= 1 \end{aligned}$$

Again, this is contradictory, as on the right hand side we have an even number ($2\cdot$(some integer)), and on the left hand side we have 1, which is an odd integer. This is not possible.

In both cases, we arrived at a contradiction, implying that our initial assumption was incorrect. Therefore, we can conclude that there are no integer solutions to $x^2 - 3 = 4y$.

2. a. Prove, using induction, that $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

   b. Prove, using induction, that $\left(\sum_{i=1}^{n} i\right)^2 = \sum_{i=1}^{n} i^3$. *(Hint: In lecture, we showed that* $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$*.)*

Solution:

a. *Base Case* $n = 1$:

$$\sum_{i=1}^{1} i^2 = 1^2 = 1; \frac{1(2)(3)}{6} = 1$$

$\therefore$, the base case holds.

Induction Hypothesis : Assume $\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$, for some arbitrary $k \in \mathbb{N}$.

Induction Step : Now, we need to show that $\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$, using the information we assumed in the hypothesis.

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1)+6(k+1))}{} \\
&= \frac{(k+1)(2k^2+7k+6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}$$

as required. Therefore, induction holds, and the statement holds true.

b. *Base Case* $n = 1$:

$$(\sum_{i=1}^{1} i)^2 = 1^2 = 1; \sum_{i=1}^{1} i^3 = 1$$

$\therefore$, the base case holds.

Induction Hypothesis : Assume $\left(\sum_{i=1}^{k} i\right)^2 = \sum_{i=1}^{k} i^3$, for some arbitrary $k \in \mathbb{N}$.

Induction Step : We need to show that $\left(\sum_{i=1}^{k+1} i\right)^2 = \sum_{i=1}^{k+1} i^3$, using the information we assumed in the hypothesis. We will rely on the identity given in the hint.

This time, we will start with the right hand side and work towards the left hand side.

$$
\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\
&= \left(\sum_{i=1}^{k} i\right)^2 + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2+4(k+1))}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 \\
&= \left(\sum_{i=1}^{k+1} i\right)^2
\end{aligned}
$$

as required. Therefore, induction holds, and the statement holds true.

3. Prove that $5|n^5 - n, \forall n \in \mathbb{N}$, using each of the following techniques:
   a. Proof by Cases *(Hint: Factor $n^5 - n$, and consider 5 possible cases for n)*
   b. Induction *(Hint: Use the fact that $(c+1)^5 = c^5 + 5c^4 + 10c^3 + 10c^2 + 5c + 1$)*

Solution:

a. We can factor $n^5 - n$ as:

$$
\begin{aligned}
n^5 - n &= n(n^4 - 1) \\
&= n(n^2 - 1)(n^2 + 1) \\
&= n(n-1)(n+1)(n^2+1)
\end{aligned}
$$

We've shown that $n^5 - n$ is the product of these four terms. If we can show that any one of these terms is a multiple of 5, we've shown that the entire product will be a multiple of 5.

Now, let's consider 5 cases for n: $n = 5k$, $n = 5k+1$, $n = 5k+2$, $n = 5k+3$ and $n = 5k+4$.

- $n = 5k$: Since n is one of the terms in the product, by default, the entire product will be a multiple of 5
- $n = 5k + 1$: Consider the term $n - 1$. If $n = 5k + 1$, then $n - 1 = 5k$, showing that the product is a multiple of 5
- $n = 5k + 2$: Consider the term $n^2 + 1$. If $n = 5k + 2$, then we have $n^2 + 1 = 25k^2 + 20k + 4 + 1 = 5(5k^2 + 4k + 1)$, which is a multiple of 5
- $n = 5k + 3$: Again, consider $n^2 + 1$. If $n = 5k + 3$, then we have $n^2 + 1 = 25k^2 + 30k + 9 + 1 = 5(5k^2 + 6k + 2)$, which is a multiple of 5
- $n = 5k + 4$: Consider the term $n + 1$. If $n = 5k + 4$, then we have $n + 1 = 5k + 4 + 1 = 5(k + 1)$, which is a multiple of 5

Therefore, in each case, we have that the product will be a multiple of 5. Therefore, we have that for all integers n, $5 | n^5 - n$.

b. *Base Case:* $n = 1$ $1^5 - 1 = 0$, and $5|0$ (since we can say $5 \cdot 0 = 0$), therefore the base case holds.

Induction Hypothesis: $n = k$ Assume $5|k^5 - k$, for some arbitrary $k \in \mathbb{N}$. We can phrase this as $k^5 - k = 5a$, for $a \in \mathbb{Z}$.

Induction Step:

We want to show $(k + 1)^5 - (k + 1)$ is a multiple of 5. To do so, first let's expand our expression, using the hint

$$\begin{aligned}
(k+1)^5 - (k+1) &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 \\
&= k^5 + 5(k^4 + 2k^3 + 2k^2 + k) - k \\
&= 5a + 5(k^4 + 2k^3 + 2k^2 - k) \text{ (using the induction hypothesis, } k^5 - k = 5a) \\
&= 5(a + k^4 + 2k^3 + 2k^2 - k)
\end{aligned}$$

Therefore, induction holds.

4. Recall the power rule for derivatives:

$$\frac{d}{dx} x^n = n x^{n-1}$$

Prove this rule for all $n \in \mathbb{N}_0$ using induction. *(Hint: You will need to use the product rule for derivatives.)*

Solution: *Base Case* $n = 0$:
$$\frac{d}{dx}x^0 = 0x^{-1} = 0$$
as required.

Induction Hypothesis $n = k$: Assume that $\frac{d}{dx}x^k = kx^{k-1}$ for some arbitrary k.

Induction Step: We will use the product rule for derivatives, as indicated in the hint.

$$\begin{aligned}
\frac{d}{dx}x^{k+1} &= \frac{d}{dx}x^k \cdot x \\
&= \left(\frac{d}{dx}x^k\right) \cdot x + x^k \cdot \left(\frac{d}{dx}x\right) \\
&= kx^{k-1} \cdot x + x^k \cdot 1 \\
&= kx^k + x^k \\
&= (k+1)x^k
\end{aligned}$$

as required. Therefore, by induction, we've proved that $\frac{d}{dx}x^n = nx^{n-1}$ for all $n \in \mathbb{N}_0$.

5. Consider the following inequality:

$$|\bigcup_{i=1}^{n} A_i| \leq \sum_{i=1}^{n} |A_i|$$

In other words, $|A_1 \cup A_2 \cup ... \cup A_n| \leq |A_1| + |A_2| + ... + |A_n|$.

Prove this using induction $\forall n \in \mathbb{N}$. *(Hint: You may need to use the Principle of Inclusion-Exclusion for two sets.)*

Solution: *Base Case* $n = 1$: $|A_1| \leq |A_1|$, therefore the base case holds.

Induction Hypothesis $n = k$ Assume that $|A_1 \cup A_2 \cup ... \cup A_k| \leq |A_1| + |A_2| + ... + |A_k|$, for some arbitrary k.

Induction Step Let $A^k = A_1 \cup A_2 \cup ... \cup A_k$. Then:

$$\begin{aligned}
|A_1 \cup A_2 \cup ... \cup A_k \cup A_{k+1}| &= |A^k \cup A_{k+1}| \\
&= |A^k| + |A_{k+1}| - |A^k \cap A_{k+1}| \quad \textit{(by the Principle of Inclusion-Exclusion)} \\
&\leq |A^k| + |A_{k+1}| \quad \textit{(since all cardinalities are non-negative)} \\
&\leq |A_1| + |A_2| + ... + |A_k| + |A_{k+1}| \quad \textit{(from the Induction Hypothesis)}
\end{aligned}$$

as required. Therefore, by induction, the original statement holds.

6. The harmonic series $H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ is known to be unbounded as $n \to \infty$. In this problem, we will use induction to prove that the harmonic series is unbounded.

   Using induction, prove that $\forall n \in \mathbb{N}, H_{2^n} \geq 1 + \frac{n}{2}$. Why does this prove that the harmonic series is unbounded?

Solution: *Base Case:* $n = 1$

$H_{2^1} = 1 + \frac{1}{2} \geq 1 + \frac{1}{2}$, as required, therefore the base case holds.

Induction Hypothesis: Assume that $H_{2^k} \geq 1 + \frac{k}{2}$ for some arbitrary integer k.

Induction Step:

$$\begin{aligned}
H_{2^{k+1}} &= H_k + \frac{1}{2^k+1} + \frac{1}{2^k+2} + ... + \frac{1}{2^{k+1}-1} + \frac{1}{2^{k+1}} \\
&\geq H_k + \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + ... + \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} \\
&= H_k + \frac{2^k}{2^{k+1}} \\
&\geq 1 + \frac{k}{2} + \frac{1}{2} \\
&= 1 + \frac{k+1}{2}
\end{aligned}$$

The inequality in the second line holds because each term $\frac{1}{2^k+1}, \frac{1}{2^k+2}, \cdots \frac{1}{2^{k+1}-1}, \frac{1}{2^{k+1}}$ is greater than or equal to the term $\frac{1}{2^{k+1}}$ (since the denominators are increasing, the fractions are decreasing, so 2^{k+1} is the largest denominator we have and thus $\frac{1}{2^{n+1}}$ is the smallest number we have).

The equality in the third line comes from the fact that there are 2^k terms of the form $\frac{1}{2^{k+1}}$. (Remember, $2^k + 2^k = 2(2^k) = 2^{k+1}$).

Therefore, by induction, we have that $H_{2^n} \geq 1 + \frac{n}{2}$.

Since there is no largest integer, there is no largest value of $1 + \frac{n}{2}$. Since $H_{2^n} \geq 1 + \frac{n}{2}$, this means that there is no largest value of H_{2^n}, meaning that the sum H_{2^n} (and also the sum H_n) does not approach a finite value as $n \to \infty$.

7. Prove that for $n \geq 3$, the sum of the interior angles of a polygon with n vertices is $180(n-2)$. *(Attempt this problem, but don't spend too long on it, and don't worry if you don't get it. Hint: Drawing pictures will help.)*

Solution: *Base Case* $n = 3$: In a triangle, by definition, the sum of the interior angles is 180.

Induction Hypothesis $n = k$: Assume that for a polygon with k vertices, the sum of the interior angles is $180(k - 2)$.

Induction Step Consider a polygon with $k + 1$ vertices. Suppose the vertices are labelled $A_1, A_2, ...A_k, A_{k+1}$. We can draw a line segment between A_1 and A_3 (or really, any A_i and A_{i+2}). This splits our figure into two sub-polygons:

- Triangle A_1, A_2, A_3, with interior angles summing to 180
- k-vertex polygon $A_3, A_4, ...A_k, A_{k+1}, A_1$, with interior angles summing to $180(k - 2)$ (from the induction hypothesis)

Then, the sum of the interior angles in the entire figure is the sum of the interior angles in the two sub-figures, which is $180+180(k-2) = 180(k-1)$, which is precisely $180((k+1)-2)$, as required.

Therefore, by induction, the statement holds.

8. In this problem, f_i will refer to the Fibonnaci sequence. This sequence is defined by $f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}, \forall n \geq 2, n \in \mathbb{N}$.

   a. Prove that $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$.

   b. Prove that $f_n > 2n$, for $n \geq 8$.

   c. Prove that $f_n \leq 2^n$.

   (Hint: You will need to use strong induction for parts b and c.)

Solution:

a. *Base Case:* $n = 1$

   $\sum_{i=1}^{1} f_i^2 = 1^2 = 1$. Also, $f_1 f_2 = 1 \cdot 1 = 1$, therefore the base case holds.

   Induction Hypothesis: Assume that $\sum_{i=1}^{k} f_i^2 = f_k f_{k+1}$ for some arbitrary value of k.

   Induction Step:

$$\begin{aligned}\sum_{i=1}^{k+1} f_i^2 &= \sum_{i=1}^{k} f_i^2 + f_{k+1}^2 \\ &= f_k f_{k+1} + f_{k+1}^2 \\ &= f_{k+1}(f_k + f_{k+1}) \\ &= f_{k+1} f_{k+2}\end{aligned}$$

   Therefore, by induction, the statement holds.

b. This problem requires us to use strong induction, as in the induction step we will expand f_{k+1} to $f_k + f_{k-1}$ and will need to use the hypothesis for both f_k and f_{k-1}.

Here, we will have two base cases, $n = 8$ and $n = 9$. $f_1, ..., f_7$. $f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5, f_6 = 8, f_7 = 13, f_8 = 21, f_9 = 34$.

Base Cases:

- $n = 8$: $f_8 = 21 > 2 \cdot 8$
- $n = 9$: $f_9 = 34 > 2 \cdot 9$

Therefore, the base cases hold.

(The reason we have two base cases: Suppose we want to prove that the statement holds for f_10. Then, we need to use the result for f_8 and f_9.)

Induction Hypothesis: Assume $f_i > 2i$ for all integers $i \in \{8, 9, ...k\}$, for some arbitrary integer k (this is where induction and strong induction differ).

Induction Step:

$$\begin{aligned} f_{k+1} &= f_k + f_{k-1} \\ &> 2k + 2(k-1) \\ &> 2k + 2 \\ &> 2(k+1) \end{aligned}$$

as required.

c. *Base Cases:*

- $n = 1$: $f_1 = 1 \leq 2^1 = 2$
- $n = 2$: $f_2 = 1 \leq 2^2 = 4$

Therefore, the base cases hold.

Induction Hypothesis: Assume that $f_i \leq 2^i$ for all integers $1 \leq i \leq k$, for some arbitrary integer k.

Induction Step:

$$\begin{aligned} f_{k+1} &= f_k + f_{k-1} \\ &\leq 2^k + 2^{k-1} \\ &\leq 2^k + 2^k \\ &\leq 2(2^k) \\ &\leq 2^{k+1} \end{aligned}$$

as required. Therefore, by induction, the statement holds.