Problem Set 5: Midterm Review

CS 198-087: Introduction to Mathematical Thinking
UC Berkeley EECS
Fall 2018

This homework will not be graded. However, it's a good idea to do as many of these problems as you can, as they will all help you in preparing for our upcoming midterm.

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1. Determine the truth value of each of the following statements.

   a. If 3 is odd, then 4 = 2 + 2.

   b. If 3 is odd, then 4 = 2 + 3.

   c. If 3 is even, then 4 = 2 + 2.

   d. If 3 is even, then 4 = 2 + 3.

   For the next few problems, assume P is true, Q is false and R is true.

   e. $(P \lor Q) \land R$

   f. $\neg Q \lor P$

   g. $(\neg P) \land (\neg Q) \land R$

   h. $P \iff Q$

   i. $(P \implies Q) \implies \neg R$

   j. $P \oplus Q \oplus R$

   k. $(P \implies Q) \oplus (\neg R)$

2. Use truth tables to prove or disprove each of the following logical equivalences. *(Hint: Recall, logically, "iff" ($\iff$) and "equivalent" ($\equiv$) mean the same thing.*

   a. $P \implies Q \equiv \neg Q \lor P$

   b. $(P \oplus Q) \equiv (P \lor Q) \land \neg(P \land Q)$

   c. $P \implies Q \equiv \neg Q \implies P$

   d. $(P \lor Q) \land R \equiv P \lor (Q \land R)$

   e. $\neg(P \lor Q) \equiv (\neg P) \land (\neg Q)$

   f. $(P \lor (P \land Q)) \iff P$ (what does this mean?)

g. $(P \wedge (P \vee Q)) \iff P$

h. $P \implies \neg(\neg Q \wedge \neg P) \equiv \text{TRUE}$

3. In each case, determine the value of the provided statement. The universe $\mathbb{U}$ is $\mathbb{Z}$.

   a. $P(17)$, where $P(x) = x \leq 20$

   b. $P(5)$, where $P(x) = (x > 20) \vee (x = 5k, k \in \mathbb{Z})$

   c. $\forall x\, ((x \geq 5) \vee (x < 5))$

   d. $\exists x\, ((x \geq 5) \vee (x < 5))$

   e. $\forall x, y\, \left(x^2 = y^2 \iff x = y\right)$

   f. $\exists x \exists y\, \left(x^2 = y^2 \iff x = y\right)$

   g. $\neg \exists x\, \left(x^2 = 0\right)$

   h. $\forall x \forall y\, (xy \geq x + y)$

   i. $\forall x \exists y (y > x)$

   j. $\forall y \neg(\exists x (y > x))$

   k. $\exists x \forall y (y > x)$

4. Use De Morgan's Laws to rewrite each of the following statements.

   a. $\neg(\exists x P(x))$

   b. $\neg(\forall x \exists y P(x, y))$

   c. $\neg(P \implies Q)$

   d. $\neg(\neg Q \vee \neg P)$

   e. $\neg(P \oplus \neg Q)$ *(Hint: Re-write $P \oplus \neg Q$, using an identity we saw in lecture and elsewhere on this homework.)*

   f. $\neg(\forall x \exists y (P(x) \vee Q(y)))$

   g. $\neg((\forall x P(x)) \vee (\exists y Q(y)))$

5. Suppose $A = \{j^2 : j \leq 5\}$, $B = \{t : t \text{ is prime}\}$, $C = \{s : s \geq 19\}$, and the universe is $\mathbb{U} = \{t : t \in \mathbb{N}_0, t \leq 25\}$.

   Determine each of the following.

   a. $A \cup B$

   b. $A^C \cup C^C$

   c. $|(A \cup B) \cap B|$

   d. $|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cup B \cup C|$

   e. $(A - B) - C$

f. $(A - B)^C \cup (B - C)^C$

6. Suppose $A = \{a, b, c\}, B = \{0, 1\}$ and $C = \{2\}$.
   a. Find $A \times B$.
   b. Find $A \times B \times C$.
   c. Find $B \times A$.
   d. Prove that if $A \times B = B \times A$, then $A = B$. *(Hint: Remember, giving an example doesn't suffice as a proof. You need to show this rigorously.)*
7. Determine whether each of the following functions is an injection, surjection, bijection, or none.
   a. $f : \mathbb{Z} \to \mathbb{Z}, f(x) = x^2 - 1$
   b. $f : \mathbb{R}_{\geq 0} \to \mathbb{R}, f(x) = \sqrt{x}$
   c. $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, f(x) = \sqrt{x}$
   d. $f : \mathbb{R} \to \mathbb{N}, f(x) = 23$
   e. $f : \mathbb{R} \to \mathbb{Z}, f(x) = \lceil x \rceil$ *(Hint: Is it possible for any function on $\mathbb{R} \to \mathbb{Z}$ to be a bijection?)*
   f. $f : \mathbb{N} \to \mathbb{Q}, f(x) = \begin{cases} \frac{1}{4}x + \frac{3}{2} & x \neq 4k, k \in \mathbb{N} \\ -\frac{1}{4}x - 1 & x = 4k, k \in \mathbb{N} \end{cases}$
8. Suppose $f(x)$ and $g(x)$ are functions.

   Prove that if $f(g(x))$ is one-to-one, then $g(x)$ is one-to-one. *(Hint: "one-to-one" is another term for "injective".)*
9. a. Prove there is no smallest positive rational number.
   b. Prove there is no largest prime number. *(Hint: All natural numbers can be written as the product of primes.)*
10. A perfect number is a positive integer n such that the sum of the factors of n that are less than n, is equal to n. For example, the factors of 6 (that are not equal to 6) are 1, 2, and 3, and 1 + 2 + 3 = 6.

    Prove that a prime number cannot be a perfect number.
11. In base 10, the integer $a_{n-1}a_{n-2}...a_1a_0$ can be written as $10^{n-1}a_{n-1} + 10^{n-2}a_{n-2} + ... + 10^2a_2 + 10^1a_1 + 10^0a_0$. For example, $427 = 4 \cdot 10^2 + 2 \cdot 10^1 + 7 = 400 + 20 + 7$.

    Suppose $n \in \mathbb{N}$.
    a. Prove that n is divisible by 3 if and only if the sum of the digits of n is divisible by 3.
    b. Prove that n is divisible by 9 if and only if the sum of the digits of n is divisible by 9.

    (Remember, to prove the statement "A if and only if B", you must prove $A \implies B$ and $B \implies A$.)

12. Prove that there is no integer $n > 3$ such that all of n, $n+2$, $n+4$ are prime. *(Hint: Break n into three cases – when it is divisible by 3, when it has a remainder of 1 when divided by 3, and when it has a remainder of 2 when divided by 3.)*

13. In any set of n numbers, there is at least one number that is less than or equal to the mean.

    a. Write this statement using propositional logic.

    b. Prove this statement.

14. Consider the series defined by $t_0 = 1$, $t_n = 2t_{n-1} + 7, \forall n \in \mathbb{N}_0$. Use induction to prove that $t_n \leq 2^{n+3} - 7$.

15. The harmonic series $H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$ is known to be unbounded as $n \to \infty$. In this problem, we will use induction to prove that the harmonic series is unbounded.

    Using induction, prove that $\forall n \in \mathbb{N}, H_{2^n} \geq 1 + \frac{n}{2}$. Why does this prove that the harmonic series is unbounded?

16. In this problem, f_i will refer to the Fibonnaci sequence. This sequence is defined by $f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}, \forall n \geq 2, n \in \mathbb{N}$.

    For parts b and c of this problem, you will need to use **strong induction**. In regular mathematical induction, in the induction hypothesis we assume that $P(k)$ holds, for some arbitrary value of k. In strong induction, instead of assuming just $P(k)$, we assume $P(0) \wedge P(1) \wedge ... P(k-1) \wedge P(k)$, i.e. that the proposition holds for all non-negative integers up to and including k. This is useful if, in our induction step, we need to assume more than just $P(k)$.

    a. Prove that $\sum_{i=1}^{n} f_i^2 = f_n f_{n+1}$.

    b. Prove that $f_n > 2n$, for $n \geq 8$.

    c. Prove that $f_n \leq 2^n$.

17. In this problem, we will prove that $3|n^3 - n$ (i.e. that $n^3 - n$ is divisible by 3) for all $n \in \mathbb{N}_0$.

    a. Prove this directly.

    b. Prove this using induction.

18. Recall, in lecture we showed that $1 + 2 + ... + n = \frac{n(n+1)}{2}$ as follows:

$$\begin{aligned}
S_n &= 1 + 2 + 3 + ... + (n-1) \\
S_n &= (n-1) + (n-2) + ... + 1 \\
2S_n &= n(n+1) \\
S_n &= \frac{n(n+1)}{2}
\end{aligned}$$

    a. An arithmetic sequence with initial term a_0 and common difference d is defined by $a_n = a_0 + (n-1)d$ for $n \in \mathbb{N}$. Prove that $\sum_{i=1}^{n} a_i = n\frac{2a_0 + (n-1)d}{2}$, using (i) induction and (ii) a direct proof similar to the one above.

b. A geometric series with initial term a and common ratio r is defined by $a_n = ar^{n-1}$ for $n \in \mathbb{N}$. Prove that $\sum_{i=1}^{n} a_i = \frac{a_0(r^n-1)}{r-1}$ using (i) induction and (ii) a direct proof similar to the one above.

19. Prove that $0.9999999.... = 1$.