

PROBLEM SET 6: COUNTING

CS 198-087: INTRODUCTION TO MATHEMATICAL THINKING
UC BERKELEY EECS
SPRING 2019

This homework is due on Sunday, April 7th, at 11:59 PM on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use \LaTeX .

1. Power Sets

Given some set S , the power set $P(S)$ of a set S is a set of all possible subsets of S . For example, if $S = \{1, 2, 3\}$, we have $P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

- If $|S| = n$, what is $|P(S)|$?
- Determine the number of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$.
- Determine the number of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that are not subsets of $\{1, 2, 3, 4\}$ or $\{3, 4, 5, 6\}$. (*Hint: You will need to use the Principle of Inclusion-Exclusion.*)

2. Counting Factors

- Determine the number of factors of 3500.
- Determine the number of factors of 3500 that are also multiples of 20.
- Determine the number of factors of 3500 that are multiples of 4 and not multiples of 20.

3. 1 to 1000, real quick

Suppose we wanted to find the number of natural numbers from 1 to 1000 (inclusive) that are multiples of 2 but not multiples of 3. We can use the following logic to proceed:

$$\begin{aligned}(\# \text{ multiples of } 2) &= (\# \text{ multiples of } 2, \text{ and } 3) + (\# \text{ multiples of } 2, \text{ and not } 3) \\ \implies (\# \text{ multiples of } 2, \text{ and not } 3) &= (\# \text{ multiples of } 2) - (\# \text{ multiples of } 2, \text{ and } 3)\end{aligned}$$

There are 500 multiples of 2 in this range — $2 \cdot 1, 2 \cdot 2, \dots, 2 \cdot 500$. A number that's a multiple of 2 and 3 is a multiple of 6, and there are 166 multiples of 6 in this range — $6 \cdot 1, 6 \cdot 2, \dots, 6 \cdot 166$. Therefore, the quantity we're looking for is $500 - 166 = 334$.

How many natural numbers from 1 to 1000 are multiples of 3 and 4, but not 5?

4. *Ball in the Family*

Suppose we have 6 basketball players who want to organize themselves into 3 basketball teams of 2 players each.

- a. Suppose we have three teams, "Team USA", "Team China" and "Team Lithuania". How many ways can these teams be formed?
 - b. Now, suppose the teams are irrelevant, and all we care about is the unique pairings themselves. How many ways can these six players be split into 3 teams? (*Hint: Is this number bigger or smaller than the number from the previous part? By what factor? What was repeated in the previous part?*)
5. The numbers 1447, 1005, and 1231 have something in common. Each of them is a four digit number that begins with 1 and has two identical digits. How many numbers like this are there? (*Hint: Consider two cases, one where the repeated digit is 1, and one where the repeated digit is not 1. The second case will then further be broken into 3 cases.*)

6. *More Fun with Cards*

How many ways can you deal 13 cards to each of 4 players so that each player gets one card of each of the 13 values (A, 2, 3, . . . , K)?

7. *Fun with Permutations*

In this question, we will deal with permutations of BERKELEY.

- a. How many permutations of BERKELEY are there?
- b. How many permutations of BERKELEY are there, where the letters BRKLY appear together, in that order?
- c. How many permutations of BERKELEY are there, where the letters BRKLY appear together, in any order?
- d. How many permutations of BERKELEY are there, where the letters BRKLY *do not* appear together, in any order?
- e. How many permutations of BERKELEY are there, where BK appear together (in any order) and LY appear together (in any order)?
- f. How many permutations of BERKELEY are there, where BK appear together (in any order) and LY do not appear together (in any order)?
- g. How many permutations of BERKELEY are there, where the letters BERK appear together, in that order? (*Hint: Notice that the E exists both in BERK and in the remaining characters. How, if at all, does this complicate things?*)
- h. How many permutations of BERKELEY are there, where the letters EEE appear together?

8. *Fun with Stars and Bars*

Determine the number of non-negative integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 19$$

subject to each of the following conditions:

- a. $x_1 \geq 0, x_2 > 4, x_3 \geq 2, x_4 \geq 1$
- b. $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, 0 \leq x_4 \leq 2$ (*Hint: Break this up into cases.*)

9. *Pascal's Identity*

As we will become familiar with next week, Pascal's Identity says the following:

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Prove this algebraically.

10. *Bonus – Largest Relatively Prime Set*

Consider the following set of sets:

$$\{S : S \subset \{1, 2, 3, 4, \dots, 30\} \wedge \forall s_i, s_j \in S, i \neq j, \gcd(s_i, s_j) = 1\}$$

This is the set of all subsets of $\{1, 2, 3, 4, \dots, 30\}$ such that each element in the subset is relatively prime to all others. One such subset is $\{2, 3, 5, 7, 11\}$, since no two numbers in this set share a factor other than 1.

Find the subset S with the largest sum. (*Hint: 30 is not in the set. 27 is.*)