Problem Set 6: Counting

CS 198-087: Introduction to Mathematical Thinking
UC Berkeley EECS
Spring 2019

This homework is due on Sunday, April 7th, at 11:59 PM on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX.

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1. *Power Sets*

   Given some set S, the power set $P(S)$ of a set S is a set of all possible subsets of S. For example, if $S = \{1, 2, 3\}$, we have $P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

   a. If $|S| = n$, what is $|P(S)|$?

   b. Determine the number of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

   c. Determine the number of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that are not subsets of $\{1, 2, 3, 4\}$ or $\{3, 4, 5, 6\}$. *(Hint: You will need to use the Principle of Inclusion-Exclusion.)*

   Solution:

   a. When creating a subset of S, for each item x_i in $S = \{x_1, x_2, ..., x_n\}$, there are 2 options — either x_i is included in the subset, or it is not. Since there are n items, and we are multiplying 2 n times, we have that $\boxed{|P(S)| = 2^n}$.

   b. This set has $n = 8$ items, so there are $2^8 = \boxed{256}$ subsets.

   c. First, we need to find the number of subsets of either $\{1, 2, 3, 4\}$ or $\{3, 4, 5, 6\}$. Then, we can subtract this quantity from the number of subsets of the larger set, to find our answer.

      Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$. Then, by inclusion-exclusion, $|A \cup B| = |A| + |B| - |A \cap B| = 2^4 + 2^4 - 2^2$ (we find that $|A \cap B| = 2^2$ because the intersection of A, B is the set $\{3, 4\}$). We want everything that is not in $|A \cup B|$, and so the quantity we are looking for is $2^8 - (2^4 + 2^4 - 2^2) = \boxed{228}$.

2. *Counting Factors*

   a. Determine the number of factors of 3500.

   b. Determine the number of factors of 3500 that are also multiples of 20.

c. Determine the number of factors of 3500 that are multiples of 4 and not multiples of 20.

Solution:

a. $3500 = 2^2 \cdot 5^3 \cdot 7$

A factor of 3500 will be of the form $2^a \cdot 5^b \cdot 7^c$. We have $2 + 1 = 3$ choices for a (0, 1 or 2), $3 + 1 = 4$ options for b and $1 + 1 = 2$ options for c, yielding $3 \cdot 4 \cdot 2 = \boxed{24}$ factors total.

b. The prime factorization of 20 is $20 = 2^2 \cdot 5$. Since our factor must be a multiple of $2^2 \cdot 5$, we are restricted in the potential values of a, b, c.

There is only one choice for a — $a = 2$. In order for a number to be a multiple of 20, it must have a factor of 2^2. There are now three options for b — $1, 2$ or 3 (b cannot be 0 now, because for a number to be a multiple of 20 it must have to have a factor of 5). Our choices for c are unchanged.

Now, the total possibilities are $1 \cdot 3 \cdot 2 = \boxed{6}$.

c. Now, we need our number to be a factor of $2^2 \cdot 5^3 \cdot 7$, a multiple of 2^2 and not a multiple of $2^2 \cdot 5$. This now again restricts our values of a, b, c.

There is still only one choice for a, $a = 2$. There is also only one possible value of b — $b = 0$. If $b \geq 1$ then $2^a \cdot 5^b \cdot 7^c$ will be a multiple of 20. Our choices for c are still unchanged.

Now, we have $1 \cdot 1 \cdot 2 = \boxed{2}$ potential numbers (these numbers are 4 and 28).

3. *1 to 1000, real quick*

Suppose we wanted to find the number of natural numbers from 1 to 1000 (inclusive) that are multiples of 2 but not multiples of 3. We can use the following logic to proceed:

$$(\text{\# multiples of 2}) = (\text{\# multiples of 2, and 3}) + (\text{\# multiples of 2, and not 3})$$
$$\implies (\text{\# multiples of 2, and not 3}) = (\text{\# multiples of 2}) - (\text{\# multiples of 2, and 3})$$

There are 500 multiples of 2 in this range – $2 \cdot 1, 2 \cdot 2, ..., 2 \cdot 500$. A number that's a multiple of 2 and 3 is a multiple of 6, and there are 166 multiples of 6 in this range — $6 \cdot 1, 6 \cdot 2, ..., 6 \cdot 166$. Therefore, the quantity we're looking for is $500 - 166 = 334$.

How many natural numbers from 1 to 1000 are multiples of 3 and 4, but not 5?

Solution:

If a number is a multiple of 3 and 4, it is a multiple of 12. Then:

(# multiples of 12, and not 5) $=$ (# multiples of 12) $-$ (# multiples of 12 and 5)

There are 83 multiples of 12 in this range $(12\cdot 1, 12\cdot 2, \ldots 12\cdot 83)$. For a number to be a multiple of 12 and 5, it is a multiple of 60, and there are 16 multiples of 60 $(60\cdot 1, 60\cdot 2, 60\cdot 3, \ldots 60\cdot 16)$ in this range. Our result is then $83 - 16 = \boxed{67}$.

4. *Ball in the Family*

Suppose we have 6 basketball players who want to organize themselves into 3 basketball teams of 2 players each.

a. Suppose we have three teams, "Team USA", "Team China" and "Team Lithuania". How many ways can these teams be formed?

b. Now, suppose the teams are irrelevant, and all we care about is the unique pairings themselves. How many ways can these six players be split into 3 teams? *(Hint: Is this number bigger or smaller than the number from the previous part? By what factor? What was repeated in the previous part?)*

Solution:

a. For Team USA, there are 6 potential players and we need to choose 2, giving us $\binom{6}{2}$ options. Then, for Team China, there are $\binom{4}{2}$ options and for Team Lithuania, $\binom{2}{2}$. Our final result is then $\boxed{\binom{6}{2}\binom{4}{2}\binom{2}{2}}$ (we multiply because we need to choose players for USA AND China AND Lithuania — "AND" means multiply).

b. The difference between this quantity and the quantity in part (a) is that the assignments to teams themselves don't matter. For example, in the first part, the assignment $\{(A, B), (C, D), (E, F)\}$ (where the first pair is for USA, second for China and third for Lithuania) would have been considered different than $\{(C, D), (E, F), (A, B)\}$, for example.

However, now all we care about is the pairings themselves. For each unique pairing, there are 3! ways to assign them to teams. This means, each unique pairing was counted 3! times in part (a), meaning we need to divide the quantity in part (a) by 3! to get our result.

Our final result is then $\boxed{\frac{\binom{6}{2}\binom{4}{2}\binom{2}{2}}{3!}}$.

5. The numbers 1447, 1005, and 1231 have something in common. Each of them is a four digit number that begins with 1 and has two identical digits. How many numbers like this are there? *(Hint: Consider two cases, one where the repeated digit is 1, and one where the repeated digit is not 1. The second case will then further be broken into 3 cases.)*

Solution: We can break this problem into two cases — where the identical digits are 1, and where they are something other than 1.

Case 1: the identical digits are 1 In this case, there are three different subcases to consider. Our number could be of the form $11xy$, or $1x1y$, or $1xy1$, where x and y are required to be different (and neither can be 1).

In each of these cases, there are 9 options for x and 8 options for y. There are three cases, giving us a total of $9 \cdot 8 \cdot 3 = 216$ options for case 1.

Case 2: identical digits are not 1 (e.g. $1xxy, 1xyx, 1yxx$)
So Now, our number can be of the form $1xxy$, $1xyx$ or $1yxx$, where again x and y are required to be different (and neither can be 1). In each of these cases, there are 9 options for x and 8 options for y. There are three cases, giving us a total of $9 \cdot 8 \cdot 3 = 216$ options for case 2.

Our total result is the sum of the options for case 1 and case 2, which is $216 \cdot 2 = \boxed{432}$.

6. *More Fun with Cards*

How many ways can you deal 13 cards to each of 4 players so that each player gets one card of each of the 13 values (A, 2, 3, . . ., K)?

Solution: There are $4!$ ways to distribute the aces to the 4 players, $4!$ ways to distribute the twos, and so on, so the number of ways to deal the cards in this manner is $4!^{13} = \boxed{24^{13}}$
.

7. *Fun with Permutations*

In this question, we will deal with permutations of BERKELEY.

a. How many permutations of BERKELEY are there?

b. How many permutations of BERKELEY are there, where the letters BRKLY appear together, in that order?

c. How many permutations of BERKELEY are there, where the letters BRKLY appear together, in any order?

d. How many permutations of BERKELEY are there, where the letters BRKLY *do not* appear together, in any order?

e. How many permutations of BERKELEY are there, where BK appear together (in any order) and LY appear together (in any order)?

f. How many permutations of BERKELEY are there, where BK appear together (in any order) and LY do not appear together (in any order)?

g. How many permutations of BERKELEY are there, where the letters BERK appear together, in that order? *(Hint: Notice that the E exists both in BERK and in the remaining* *characters. How, if at all, does this complicate things?)*

h. How many permutations of BERKELEY are there, where the letters EEE appear together?

Solution:

a. There are 8 characters in total. The character E repeats 3 times. We start with $8!$ and divide by the number of ways to arrange the repeated Es, yielding $\boxed{\frac{8!}{3!} = 6720}$.

b. We can block together BRKLY as one character. We then have 4 characters (BRKLY, E, E, E), of which the triple Es repeat. We start with $4!$ and divide by the movement of the repeated Es, giving our final result $\boxed{\frac{4!}{3!} = 4}$.

c. Now, we multiply our previous result by $5!$, to account for all permutations of BRKLY within its block. This gives $\boxed{\frac{4!5!}{3!} = 480}$.

d. Note, in any permutation of BERKELEY, either the characters BRKLY appear together (in any order), or they do not. We have the total number of permutations, and the number of permutations where they appear together (in any order). By subtracting the latter from the former, we'll be left with the number of permutations of BERKELEY without BRKLY together. This is given by $6720 - 480 = \boxed{6240}$.

e. We block together BK and LY together, effectively giving us 6 characters (BK, LY, E, E, E, R). We start with $\frac{6!}{3!}$, which considers the number of permutations of these 6 characters factoring in the repeated Es. However, we were told that the characters BK could appear in any order (i.e. we need to consider BK and KB), as with LY and YL. This means we need to multiply our current result by $2!2!$, giving our final result as $\boxed{\frac{6!2!2!}{3!} = 480}$.

f. To find our result, we can find the total number of permutations with BK together, and subtract the number of permutations with BK together and LY together.

   We've already identified the last quantity, in the previous part of this question. We now need to find the number of permutations of BERKELEY with BK together in any order, which is $\frac{7!2!}{3!} = 1680$. This gives our final answer as $\boxed{1680 - 480 = 1200}$.

g. By blocking together BERK, we now have 5 characters. We only need to consider the repeated Es outside of BERK, as the one inside has already been ”locked in” (to see this, you could just think of BERK as being the character X, and looking at the permutations of XELEY). Our result is then $\boxed{\frac{5!}{2!} = 60}$.

h. We group all three Es together as a single character, meaning we now have 6 characters. However, we don't need to adjust for the repeated Es, as they are already ”locked in” we've forced a particular ordering of them. Again, you could think of this as replacing EEE with X, and looking at all permutations of BRKLYX. This means we have $\boxed{6! = 720}$ total permutations of this form.

8. *Fun with Stars and Bars*

Determine the number of non-negative integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 19$$

subject to each of the following conditions:

a. $x_1 \geq 0, x_2 > 4, x_3 \geq 2, x_4 \geq 1$

b. $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, 0 \leq x_4 \leq 2$ *(Hint: Break this up into cases.)*

Solution:

a. Let's do a change of variables into some x_1', x_2', x_3' and x_4' such that the only conditions on x_i' are that $x_i' \geq 0$. Then, we can use the standard solution to stars and bars to determine the number of solutions.

Notice that the condition $x_2 > 4$ is equivalent to $x_2 \geq 5$.

$$\begin{aligned} x_1' &= x_1 \\ x_2' &= x_2 - 5 \\ x_3' &= x_3 - 2 \\ x_4' &= x_4 - 1 \end{aligned}$$

Then, we have

$$\begin{gathered} x_1 + x_2 + x_3 + x_4 = 19 \\ x_1 + (x_2 - 5) + (x_3 - 2) + (x_4 - 1) = 19 - 5 - 2 - 1 \\ x_1' + x_2' + x_3' + x_4' = 11 \end{gathered}$$

Now, we can treat our problem as if we have 11 stars and 3 bars, yielding

$\boxed{\binom{11+3}{3} = \binom{14}{3}}$ integer solutions to this equation.

b. Notice the inequality on x_4 is now an upper bound, as opposed to a lower bound. This means there are three cases to consider: $x_4 = 0, x_4 = 1$ and $x_4 = 2$.

Case 1: $x_4 = 0$

We are now looking at the number of non-negative integer solutions to $x_1 + x_2 + x_3 = 19$, which is given by $\binom{21}{2}$.

Case 2: $x_4 = 1$

We are now looking at $x_1 + x_2 + x_3 + 1 = 19$, i.e. $x_1 + x_2 + x_3 = 18$, which has $\binom{20}{2}$ non-negative integer solutions.

Case 3: $x_4 = 2$

Following the pattern, there are $\binom{19}{2}$ non-negative integer solutions to $x_1 + x_2 + x_3 = 17$.

Our total number of solutions is then

$\boxed{\binom{19}{2} + \binom{20}{2} + \binom{21}{2} = 571}$.

9. *Pascal's Identity*

As we will become familiar with next week, Pascal's Identity says the following:

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Prove this algebraically.

Solution:

$$\binom{n}{k} + \binom{n}{k+1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!}$$

$$= \frac{n!(k+1)}{(k+1)!(n-k)!} + \frac{n!(n-k)}{(k+1)!(n-k)!}$$

$$= \frac{n!(k+1+n-k)}{(k+1)!(n+k)!} = \frac{(n+1)!}{(k+1)!(n+k)!}$$

$$= \binom{n+1}{k+1}$$

10. *Bonus – Largest Relatively Prime Set*

Consider the following set of sets:

$$\{S : S \subset \{1, 2, 3, 4, ..., 30\} \land \forall s_i, s_j \in S, i \neq j, \text{gcd}(s_i, s_j) = 1\}$$

This is the set of all subsets of $\{1, 2, 3, 4, ..., 30\}$ such that each element in the subset is relatively prime to all others. One such subset is $\{2, 3, 5, 7, 11\}$, since no two numbers in this set share a factor other than 1.

Find the subset S with the largest sum. *(Hint: 30 is not in the set. 27 is.)*

Solution: The set is given by $\boxed{\{1, 11, 13, 17, 19, 23, 25, 27, 28, 29\}}$. Read this link for details, or feel free to bring it up in OH/on Piazza/over email!