Problem Set 7: Counting

CS 198-087: Introduction to Mathematical Thinking
UC Berkeley EECS
Fall 2018

This homework is due on Wednesday, October 31st, at 11:59PM, on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX.

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1. *Power Sets*

   Given some set S, the power set $P(S)$ of a set S is a set of all possible subsets of S. For example, if $S = \{1, 2, 3\}$, we have $P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

   a. If $|S| = n$, what is $|P(S)|$?

   b. Determine the number of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$.

   c. Determine the number of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that are not subsets of $\{1, 2, 3, 4\}$ or $\{3, 4, 5, 6\}$. *(Hint: You will need to use the Principle of Inclusion-Exclusion.)*

2. *Counting Factors*

   a. Determine the number of factors of 3500.

   b. Determine the number of factors of 3500 that are also multiples of 12.

   c. Determine the number of factors of 3500 that are multiples of 4 and not multiples of 20.

3. *1 to 1000, real quick*

   Suppose we wanted to find the number of natural numbers from 1 to 1000 (inclusive) that are multiples of 2 but not multiples of 3. We can use the following logic to proceed:

$$\text{(\# multiples of 2)} = \text{(\# multiples of 2, and 3)} + \text{(\# multiples of 2, and not 3)}$$
$$\implies \text{(\# multiples of 2, and not 3)} = \text{(\# multiples of 2)} - \text{(\# multiples of 2, and 3)}$$

   There are 500 multiples of 2 in this range – $2 \cdot 1, 2 \cdot 2, ..., 2 \cdot 500$. A number that's a multiple of 2 and 3 is a multiple of 6, and there are 166 multiples of 6 in this range — $6 \cdot 1, 6 \cdot 2, ..., 6 \cdot 166$. Therefore, the quantity we're looking for is $500 - 166 = 334$.

   How many natural numbers from 1 to 1000 are multiples of 3 and 4, but not 5?

4. *Ball in the Family*

   Suppose we have 6 basketball players who want to organize themselves into 3 basketball teams of 2 players each.

   a. Suppose we have three teams, "Team USA", "Team China" and "Team Lithuania". How many ways can these teams be formed?

   b. Now, suppose the teams are irrelevant, and all we care about is the unique pairings themselves. How many ways can these six players be split into 3 teams? *(Hint: Is this number bigger or smaller than the number from the previous part? By what factor? What was repeated in the previous part?)*

5. The numbers 1447, 1005, and 1231 have something in common. Each of them is a four digit number that begins with 1 and has two identical digits. How many numbers like this are there? *(Hint: Consider two cases, one where the repeated digit is 1, and one where the repeated digit is not 1. The second case will then further be broken into 3 cases.)*

6. *More Fun with Cards*

   How many ways can you deal 13 cards to each of 4 players so that each player gets one card of each of the 13 values (A, 2, 3. . .K)?

7. *Fun with Permutations*

   In this question, we will deal with permutations of BERKELEY.

   a. How many permutations of BERKELEY are there?

   b. How many permutations of BERKELEY are there, where the letters BRKLY appear together, in that order?

   c. How many permutations of BERKELEY are there, where the letters BRKLY appear together, in any order?

   d. How many permutations of BERKELEY are there, where the letters BRKLY *do not* appear together, in any order?

   e. How many permutations of BERKELEY are there, where BK appear together (in any order) and LY appear together (in any order)?

   f. How many permutations of BERKELEY are there, where BK appear together (in any order) and LY do not appear together (in any order)?

   g. How many permutations of BERKELEY are there, where the letters BERK appear together, in that order? *(Hint: Notice that the E exists both in BERK and in the remaining characters. How, if at all, does this complicate things?)*

   h. How many permutations of BERKELEY are there, where the letters EEE appear together?

8. *Pascal's Theorem*

   As we will become familiar with next week, Pascal's theorem says the following:

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Prove this algebraically.

9. *Bonus – Largest Relatively Prime Set*

   Consider the following set of sets:

   $$\{S : S \subset \{1, 2, 3, 4, ..., 30\} \wedge \forall s_i, s_j \in S, i \neq j, \text{gcd}(s_i, s_j) = 1\}$$

   This is the set of all subsets of $\{1, 2, 3, 4, ..., 30\}$ such that each element in the subset is relatively prime to all others. One such subset is $\{2, 3, 5, 7, 11\}$, since no two numbers in this set share a factor other than 1.

   Find the subset S with the largest sum. *(Hint: 30 is not in the set. 27 is.)*