Problem Set 7: Binomial Theorem and Vieta's Formulas Basics

CS 198-087: Introduction to Mathematical Thinking
UC Berkeley EECS
Spring 2019

This homework is due on Sunday, April 14th, at 11:59 PM on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX.

---

1. *Binomial Theorem — General Term*

   Let $g(x) = (2x^5 - 3x^2)^7$.

   a. What is the sum of the coefficients of the expansion of $g(x)$?

   b. Find the general term of the expansion of $g(x)$.

   c. What is the coefficient on x^{20}?

   d. What is the coefficient on x^{18}?

Solution:

a. To find the sum of the coefficients in an expansion, we set the values of all variables to 1. In our case, $g(1) = (2 \cdot 1^5 - 3 \cdot 1^2)^7 = (2-3)^7 = (-1)^7 = -1$, meaning the sum of the coefficients of this expansion is $\boxed{-1}$.

b.

$$\begin{aligned} t_k &= \binom{7}{k}(2x^5)^{7-k}(-3x^2)^k \\ &= \boxed{(-1)^k \binom{7}{k} 2^{7-k} 3^k x^{35-3k}} \end{aligned}$$

as required.

c. To find the coefficient on x^{20}, we set the exponent $35 - 3k = 20$, and solve to find $k = 5$. We then substitute $k = 5$ into the general term, yielding $t_5 = (-1)^5 \binom{7}{5} 2^{7-5} 3^5 x^{20}$, meaning the coefficient on x^{20} is $\boxed{(-1)^5 \binom{7}{5} 2^{7-5} 3^5 = -20412}$

d. Solving $35 - 3k = 18$ yields $k = \frac{17}{3}$. Since k is our index for term number in the binomial expansion, there is no meaning for non-integer values of k. This means x^{18} does not appear in the expansion of $g(x)$, meaning the coefficient is $\boxed{0}$.

2. *Approximations with the Binomial Theorem*

Use the first three terms of the binomial expansion to approximate each of the following values. Use a calculator to simplify immediate steps if need be, but only when absolutely necessary.

Compare your results with the true values.

a. 5.02^3

b. $31^{-\frac{1}{5}}$

Solution:

a. In part(a), our exponent is an integer, so the preface to this problem really doesn't apply. We can proceed as normal.

$$\begin{aligned} 5.02^3 &= (5 + 0.02)^3 \\ &= 5^3 + \binom{3}{1}5^2 \cdot 0.02 + \binom{3}{2}5^1 \cdot 0.02^2 + \binom{3}{3}0.02^3 \\ &= 125 + 75 \cdot 0.02 + 15 \cdot 0.0004 + 0.000008 \\ &= \boxed{126.506008} \end{aligned}$$

Here, this value is exact, because we completed the binomial expansion.

b. We'll use the fact that we know that $9^{\frac{1}{2}} = 3$.

$$\begin{aligned} 9.08^{\frac{1}{2}} &= (9 + 0.08)^{\frac{1}{2}} \\ &= 9^{\frac{1}{2}} + (\frac{1}{2})9^{-\frac{1}{2}}0.08 + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}9^{-\frac{3}{2}}0.08^2 \\ &= 3 + \frac{1}{2} \cdot \frac{1}{3} \cdot 0.08 - \frac{1}{8} \cdot \frac{1}{27} \cdot 0.08^2 \\ &= \boxed{3.0133037} \end{aligned}$$

Using a calculator gives $9.08^{1/2} = 3.0133038$, which is very close to our result from using just 3 binomial expansion terms.

Note: This expansion is infinite. For integer exponents, we follow the sequence $\binom{n}{0}$, $\binom{n}{1}$, ... which ends at $\binom{n}{n}$. However, for fractions, the sequence 1, n, $\frac{n(n-1)}{2}$, $\frac{n(n-1)(n-2)}{6}$, $\frac{n(n-1)(n-2)(n-3)}{24}$, ... has no end (as no term $n - i$ will ever be equal to 1).

Additionally, if this sequence had some end, it would imply $9.08^{1/2}$ is rational (which it is not).

c. Now, we have a negative exponent. This doesn't change our process, though! We use the fact that $32^{-\frac{1}{5}} = \frac{1}{2}$. Then:

$$\begin{aligned}
31^{-\frac{1}{5}} &= (32-1)^{-\frac{1}{5}} \\
&= 32^{-\frac{1}{5}} + (-\frac{1}{5}) \cdot 32^{-\frac{6}{5}}(-1) + (-\frac{1}{5})(-\frac{6}{5})32^{-\frac{11}{5}}(-1)^2 \\
&= \frac{1}{2} + (\frac{1}{5})\frac{1}{2^6} - \frac{6}{25} \cdot \frac{1}{2^{11}} \\
&= \boxed{0.5030078125}
\end{aligned}$$

Here, our solution isn't as accurate with just 3 terms, as a calculator tells us $31^{-\frac{1}{5}} = 0.503184971$. However, we did identify the value correctly to the first three decimal places, and with more terms we would converge on the solution.

3. *Sums of Coefficients*

a. Three roots of $x^4 + ax^2 + bx + c = 0$ are $9, -3$ and 2. Determine $a + b + c$. *(Hint: What is the coefficient of x^3?)*

b. Suppose $P(x)$ is a polynomial such that

$$x^{23} + 23x^{17} - 18x^{16} - 24x^{15} + 108x^{14} = (x^4 - 3x^2 - 2x + 9)P(x)$$

Determine the sum of the coefficients of $P(x)$.

Solution:

a. Suppose $r_1 = 9$, $r_2 = -3$, and $r_3 = 2$. We know that our polynomial is of degree 4, so there must be some r_4.

Given that the coefficient on x^3 is 0, this means that the sum of the roots is $-0 = 0$. Using this, we can find r_4, since $r_1 + r_2 + r_3 + r_4 = 0 \implies 9 - 3 + 2 + r_4 = 0 \implies r_4 = -8$.

Then, a is the sum of the product of all possible pairs of roots (of which there are $\binom{4}{2}$), b is the negative of the sum of the product of all possible triplets of the roots, and c is the product of the roots.

$$\begin{aligned} a &= r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 \\ &= r_1(r_2 + r_3 + r_4) + r_2(r_3 + r_4) + r_3r_4 \\ &= 9(0 - 9) - 3(2 - 8) + 2(-8) \\ &= -81 + 18 - 16 \\ &= -79 \end{aligned}$$

$$\begin{aligned} -b &= r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 \\ &= 9(-3)(2) + 9(-3)(-8) + 9(2)(-8) + (-3)(2)(-8) \\ &= 9(-6 + 24 - 16) + 48 \\ &= 18 + 48 \\ &= 66 \\ \implies b &= -66 \end{aligned}$$

$$\begin{aligned} c &= r_1r_2r_3r_4 \\ &= 9(-3)(2)(-8) \\ &= 432 \end{aligned}$$

Then, $a + b + c = -79 - 66 + 432 = \boxed{287}$, and our original polynomial was $x^4 - 79x^2 - 66x + 432$. (You can use Wolfram Alpha to confirm this result.)

b. Recall, to find the sum of the coefficients of $P(x)$, we want to determine $P(1)$.

$$\begin{aligned} x^{23} + 23x^{17} - 18x^{16} - 24x^{15} + 108x^{14} &= (x^4 - 3x^2 - 2x + 9)P(x) \\ 1^{23} + 23 \cdot 1^{17} - 18 \cdot 1^{16} - 24 \cdot 1^{15} + 108 \cdot 1^{14} &= (1^4 - 3 \cdot 1^2 - 2 \cdot 1 + 9)P(1) \\ 1 + 23 - 18 - 24 + 108 &= (1 - 3 - 2 + 9)P(1) \\ 90 &= 5P(1) \\ \implies P(1) &= 18 \end{aligned}$$

Therefore, the sum of the coefficients of $P(x)$ is 18.

Note: Alternatively, you could have used some method of polynomial division to explicitly determine $P(x)$, but that is not necessary. However, we can use a site such as Wolfram Alpha to determine $P(x)$ to verify our result.