Problem Set 8: Polynomials, Review

CS 198-087: Introduction to Mathematical Thinking
UC Berkeley EECS
Spring 2019

This homework is due on Sunday, April 28th, at 11:59 PM on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX. *Note: Problem 5 is unrelated to the content from the last few weeks, but it is still a good problem to attempt.*

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1. *Determining Coefficients*

   a. Determine the coefficient of x^{50} in the expansion of

   $$(x+1)^{1000} + x(x+1)^{999} + x^2(x+1)^{998} + ... + x^{999}(x+1) + x^{1000}$$

   (Hint: You may need to use the Hockey Stick identity.)

   b. Determine the coefficient of x^3 in the expansion of

   $$(x^2 + x - 5)^3$$

2. *Evaluating Sums*

   Evaluate the sum

   $$\sum_{k=0}^{n} k\binom{n}{k}(-1)^{k-1}3^{n-k}$$

   (Hint: Replace -1 with a variable. What is this sum the derivative of?)

3. *Product of Multiple Binomial Expansions*

   Let's explore another application of the binomial theorem. Let $f(x,y) = (2x-3y)^5$ and $g(x,y) = (x^3 - 3xy^2)^9$.

   a. Find the general terms of both $f(x,y)$ and $g(x,y)$. Use the index variable k for $f(x,y)$ and i for $g(x,y)$.

   b. Find the combined general term, that is, find the general term of $f(x,y) \cdot g(x,y)$. It will be of the form $t_{k,i} = \binom{5}{k}\binom{9}{i}...$

   c. Find the sum of the coefficients of the product $f(x,y) \cdot g(x,y)$.

d. Determine all terms containing x^{14} in the expansion of $f(x,y) \cdot g(x,y)$.

4. *Vieta's Practice*

   a. Let $f(x) = 5x^3 - 4x^2 + 16x - 3$ have roots r_1, r_2, r_3. Find $r_1^2 r_2 r_3 + r_1 r_2^2 r_3 + r_1 r_2 r_3^2$.

   b. Find all values of m such that $2x^2 - mx - 8$ has roots that differ by $m - 1$.

   c. Suppose a and b satisfy $x^2 - mx + 2 = 0$. Also, suppose $a + \frac{1}{b}$ and $b + \frac{1}{a}$ satisfy $x^2 - px + q = 0$. Determine q in terms of a, b, p, m.

5. *Triangular Numbers*

   Triangular numbers are numbers in the set $\{1, 3, 6, 10, 15, 21, ...\}$. The n-th triangular number, for $n \geq 1$, is given by $\binom{n+1}{2}$.

   a. Determine a closed form expression for

   $$1 + 3 + 6 + 10 + ... + \binom{n+1}{2} = \sum_{k=2}^{n+1} \binom{k}{2}$$

   using the fact that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. It should be a cubic polynomial in n.

   b. Prove your closed form expression holds using induction.