Problem Set 8: Stars and Bars, Pascal's Triangle, Combinatorial Proofs, Binomial Theorem

CS 198-087: Introduction to Mathematical Thinking
UC Berkeley EECS
Fall 2018

This homework is due on Wednesday, November 7th, at 11:59PM, on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX.

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1. *Fun with Stars and Bars*

   Determine the number of non-negative integer solutions to the equation

   $$x_1 + x_2 + x_3 + x_4 = 19$$

   subject to each of the following conditions:

   a. $x_1 \geq 0, x_2 > 4, x_3 \geq 2, x_4 \geq 1$

   b. $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, 0 \leq x_4 \leq 2$ *(Hint: Break this up into cases.)*

2. *Combinatorial Proof — n^2*

   In this problem, we'll investigate the identity

   $$n^2 = \binom{n}{2} + \binom{n+1}{2}$$

   a. Give an interpretation of this identity in terms of Pascal's Triangle.

   b. Give a combinatorial proof of the identity $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n+1}{k+1}$.

   c. Give a combinatorial proof of $n^2 = \binom{n}{2} + \binom{n+1}{2}$. You will need to use the identity above as an intermediate step.

3. *Hockey Stick Theorem*

   The "Hockey Stick Theorem" is shown on Pascal's Triangle below:

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}
$$

In the above example, $56 = 1 + 3 + 6 + 10 + 15 + 21$. Algebraically, the Hockey Stick Theorem states for any integers n, r such that $n \geq r$:

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}$$

a. Prove this statement using induction.

b. Prove this statement algebraically, using Pascal's Identity.

c. Prove this statement using a combinatorial proof.

4. *Optional — More Practice with Combinatorial Proofs*

   Give combinatorial proofs for each of the following statements.

   a. $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$

   b. $\binom{2n}{n} = 2\binom{2n-1}{n-1}$ *(Hint: Notice,* $\binom{2n-1}{n-1} = \binom{2n-1}{n}$*)*

5. *Binomial Theorem — General Term*

   Let $g(x) = (2x^5 - 3x^2)^7$.

   a. What is the sum of the coefficients of the expansion of $g(x)$?

   b. Find the general term of the expansion of $g(x)$.

   c. What is the coefficient on x^{20}?

   d. What is the coefficient on x^{18}?

6. *Approximations with the Binomial Theorem*

   Use the first three terms of the Binomial Theorem to approximate the following values:

   The combinatorial term $\binom{n}{k}$ is only valid when n and k are integers, since factorials (as we've seen them so far) are only defined for integers.

   However, we can rewrite terms of the form $\binom{n}{k}$ so that they don't involve factorials, e.g.

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$
$$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6}$$

This allows us to approximate numbers raised to non-integer powers. For example, suppose we want to approximate $8.03^{\frac{1}{3}}$. Using just the first few terms of the binomial expansion yield:

$$\begin{aligned}
8.03^n &= (8 + 0.03)^n \\
&= 8^n + n \cdot 8^{n-1} \cdot 0.03 + \frac{n(n-1)}{2} 8^{n-2} \cdot 0.03^2 \\
&= 8^{\frac{1}{3}} + \frac{1}{3} \cdot 8^{-\frac{2}{3}} \cdot 0.03 + \frac{\frac{1}{3} \cdot (-\frac{2}{3})}{2} 8^{-\frac{5}{3}} \cdot 0.03^2 \\
&= 2.002496875
\end{aligned}$$

With a calculator, we have that this expansion is equal to 2.002496875, which is very close to the true value of 2.00249688.

Use the first three terms of the binomial expansion to approximate each of the following values. Use a calculator if need be, but only when absolutely necessary.

a. 5.02^3

b. $9.08^{\frac{1}{2}}$

c. $31^{-\frac{1}{5}}$