Problem Set 8: Polynomials, Review

CS 198-087: Introduction to Mathematical Thinking
UC Berkeley EECS
Spring 2019

This homework is due on Sunday, April 28th, at 11:59 PM on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX. *Note: Problem 5 is unrelated to the content from the last few weeks, but it is still a good problem to attempt.*

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1. *Determining Coefficients*

   a. Determine the coefficient of x^{50} in the expansion of

   $$(x+1)^{1000} + x(x+1)^{999} + x^2(x+1)^{998} + ... + x^{999}(x+1) + x^{1000}$$

   (Hint: You may need to use the Hockey Stick identity.)

   b. Determine the coefficient of x^3 in the expansion of

   $$(x^2 + x - 5)^3$$

Solution:

a. First, note that our sum can be written $\sum_{k=0}^{1000} x^k(x+1)^{1000-k}$, and note that the smallest degree term of each term in our sum is k. For example, $x^{33}(x+1)^{967}$ will only consist of terms of degree 33 or higher. Since we're trying to find the coefficient on x^{50}, we only need to look at the sum from $k = 0$ to 50, as no terms after that point will contain a x^{50} and thus will not change the coefficient.

$$(x+1)^{1000} + x(x+1)^{999} + x^2(x+1)^{998} + ... + x^{50}x^{950}$$

Now, we need to consider the coefficient of x^{50} in each of our (51) sub-polynomials. Note, the coefficient of x^{50} in $x(x+1)^{999}$ is really the coefficient of x^{49} in x^{999}, as the multiplication by x "shifts" everything up one degree. Similarly, the coefficient of x^{50} in $x^{23}(x+1)^{977}$ is really the coefficient of x^{27} in $(x+1)^{977}$, and more generally, the coefficient of x^{50} in $x^k(x+1)^{1000-k}$ is the coefficient of x^{50-k} in $(x+1)^{1000-k}$. The sum we are now trying to evaluate is

$$\begin{aligned}
&\text{coefficient of } x^{50} \text{ in } (x+1)^{1000} \\
&+\text{coefficient of } x^{49} \text{ in } (x+1)^{999} \\
&+\text{coefficient of } x^{48} \text{ in } (x+1)^{998} \\
&+\vdots \\
&+\text{coefficient of } x^{1} \text{ in } (x+1)^{951} \\
&+\text{coefficient of } x^{0} \text{ in } (x+1)^{950}
\end{aligned}$$

Now, note that the coefficient of x^i in $(x+1)^n$ is $\binom{n}{n-i}$ (it is also $\binom{n}{i}$, but the former form makes our sum significantly easier to interpret). So, our problem now boils down to evaluating

$$\binom{1000}{950} + \binom{999}{950} + \ldots + \binom{951}{950} + \binom{950}{950} = \sum_{k=0}^{50} \binom{1000-k}{950}$$

Using the Hockey Stick identity we discussed in class, this sum evaluates to $\boxed{\binom{1001}{951}}$, which is the coefficient on x^{50} in the original polynomial, as required.

This is a very challenging problem!

b. Recall, the general term of this expansion will be of the form

$$t_{a,b,c} = \frac{3!}{a!b!c!}(x^2)^a x^b (-5)^c = \frac{3!}{a!b!c!} x^{2a+b}(-5)^c$$

Now, we need to set the exponent on x, $2a+b$, to 3, and solve for all possible pairs of a, b, c such that $a+b+c=3$.

This is achieved by the triplets $(0,3,0)$ and $(1,1,1)$. Then:

$$t_{a=0,b=3,c=0} = \frac{3!}{0!3!0!} x^3(-5)^0 = x^3$$

$$t_{a=1,b=1,c=1} = \frac{3!}{1!1!1} x^3(-5)^1 = -30x^3$$

Therefore, the coefficient on x^3 in this expansion is $1 - 30 = \boxed{-29}$.

We can consult WolframAlpha to verify our result.

2. *Evaluating Sums*

Evaluate the sum

$$\sum_{k=0}^{n} k\binom{n}{k}(-1)^{k-1}3^{n-k}$$

(Hint: Replace -1 with a variable. What is this sum the derivative of?)

Solution:

Let $x = -1$. Then, our sum is of the form

$$\sum_{k=0}^{n}\binom{n}{k}kx^{k-1}3^{n-k}$$

Recall, $\frac{d}{dx}x^k = kx^{k-1}$, and we have a factor of kx^{k-1} in our sum. Additionally, everything else in our sum is a constant with respect to x. We can then say, using the binomial theorem:

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}kx^{k-1}3^{n-k} &= \frac{d}{dx}\sum_{k=0}^{n}\binom{n}{k}3^{n-k}x^k \\ &= \frac{d}{dx}(3+x)^n \\ &= n(3+x)^{n-1} \\ &= \boxed{n(2)^{n-1}}\end{aligned}$$

(Note: The question really should have read "simplify", not "evaluate", since a value of n isn't specified.)

3. *Product of Multiple Binomial Expansions*

   Let's explore another application of the binomial theorem. Let $f(x, y) = (2x - 3y)^5$ and $g(x, y) = (x^3 - 3xy^2)^9$.

   a. Find the general terms of both $f(x, y)$ and $g(x, y)$. Use the index variable k for $f(x, y)$ and i for $g(x, y)$.

   b. Find the combined general term, that is, find the general term of $f(x, y) \cdot g(x, y)$. It will be of the form $t_{k,i} = \binom{5}{k}\binom{9}{i}...$

   c. Find the sum of the coefficients of the product $f(x, y) \cdot g(x, y)$.

   d. Determine all terms containing x^{14} in the expansion of $f(x, y) \cdot g(x, y)$.

Solution:

a.
$$t_k = (-1)^k \binom{5}{k} 2^{5-k} 3^k x^{5-k} y^k$$
$$t_i = (-1)^i \binom{9}{i} 3^i x^{27-2i} y^{2i}$$

b.
$$t_{k,i} = (-1)^{k+i} \binom{5}{k} \binom{9}{i} 2^{5-k} 3^{k+i} x^{32-(k+2i)} y^{k+2i}$$

Note the relationship between the exponent on x and the exponent on y.

c. To get the sum of coefficients, we set $x = y = 1$.

$$f(1,1)g(1,1) = (-1)^5(1-3)^9 = \boxed{512}$$

d. We need to find all solutions to $32 - (k + 2i) = 14$ — in other words, to $k + 2i = 18$ — with the constraints $k \in [0, 5]$ and $i \in [0, 9]$.

This is attained by:

- $k = 0, i = 9$
- $k = 2, i = 8$
- $k = 4, i = 7$

Then:

$$t_{k=0,i=9} = -\binom{5}{0}\binom{9}{9} 2^5 3^9 x^{14} y^{18}$$

$$t_{k=2,i=8} = \binom{5}{2}\binom{9}{8} 2^3 3^{10} x^{14} y^{18}$$

$$t_{k=4,i=7} = -\binom{5}{4}\binom{9}{7} 2^1 3^{11} x^{14} y^{18}$$

Note, the terms that contain x^{14} are exactly the terms that contain y^{18} (in other words, all three of the above are "like terms".) The coefficient of these terms is then

$$-2^5 \cdot 3^9 + 10 \cdot 9 \cdot 2^3 \cdot 3^{10} - 5 \cdot 36 \cdot 2^1 \cdot 3^{11} = -21887496$$

Therefore, the only term containing x^{14} in the above expansion is $\boxed{-21887496x^{14}y^{18}}$. You can confirm this result using WolframAlpha.

4. *Vieta's Practice*

   a. Let $f(x) = 5x^3 - 4x^2 + 16x - 3$ have roots r_1, r_2, r_3. Find $r_1^2 r_2 r_3 + r_1 r_2^2 r_3 + r_1 r_2 r_3^2$.

   b. Find all values of m such that $2x^2 - mx - 8$ has roots that differ by $m - 1$.

   c. Suppose a and b satisfy $x^2 - mx + 2 = 0$. Also, suppose $a + \frac{1}{b}$ and $b + \frac{1}{a}$ satisfy $x^2 - px + q = 0$. Determine q in terms of a, b, p, m.

Solution:

a. We can factor $r_1^2 r_2 r_3 + r_1 r_2^2 r_3 + r_1 r_2 r_3^2$ as $r_1 r_2 r_3 (r_1 + r_2 + r_3)$. Then, from Vieta's, we know that $r_1 + r_2 + r_3 = -\frac{-4}{5} = \frac{4}{5}$ and $r_1 r_2 r_3 = -\frac{-3}{5} = \frac{3}{5}$. Then, the quantity we're looking for is $\boxed{\frac{12}{25}}$.

b. Suppose r_1, r_2 are the roots of this equation.

   Then, since we have the equation $2x^2 - mx - 8$, we know that $r_1 + r_2 = -\frac{-m}{2} = \frac{m}{2}$, $r_1 r_2 = -4$, and we want $r_1 - r_2 = m - 1$ (we could alternatively say $r_2 - r_1 = m - 1$: it wouldn't change anything). Solving using the first and third equations, we can find the following expressions for r_1, r_2 in terms of m:

$$r_1 = \frac{3}{4}m - 1$$
$$r_2 = \frac{1}{2}m - r_1 = -\frac{1}{4}m + \frac{1}{2}$$

   Then, since we have that $r_1 r_2 = -4$, we can multiply our expressions for r_1, r_2 and solve for m.

$$\begin{aligned} r_1 r_2 &= -4 \\ \left(\frac{3}{4}m - 1\right)\left(-\frac{1}{4}m + \frac{1}{2}\right) &= -4 \\ (3m-2)(m-2) &= 64 \\ 3m^2 - 8m - 60 = (m-6)(3m+10) &= 0 \end{aligned}$$

   This tells us that the possible values for m are $\boxed{6, -\frac{10}{3}}$.

c. Since a, b are roots of $x^2 - mx + 2$, we know that $a + b = m$ and $ab = 2$.

   Since $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are roots of $x^2 - px + q$, we know that $a + \frac{1}{b} + b + \frac{1}{a} = p$ and $\left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) = q$.

Expanding out the expression for q:

$$q = \left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) ab + 1 + 1 + \frac{1}{ab}$$

Since we know that $ab = 2$, we can actually determine a numerical value for q:

$$q = 2 + 1 + 1 + \frac{1}{2} = \boxed{\frac{9}{2}}$$

5. *Triangular Numbers*

Triangular numbers are numbers in the set $\{1, 3, 6, 10, 15, 21, ...\}$. The n-th triangular number, for $n \geq 1$, is given by $\binom{n+1}{2}$.

a. Determine a closed form expression for

$$1 + 3 + 6 + 10 + ... + \binom{n+1}{2} = \sum_{k=2}^{n+1} \binom{k}{2}$$

using the fact that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$. It should be a cubic polynomial in n.

b. Prove your closed form expression holds using induction.

Solution:

a.

$$\begin{aligned}
\sum_{k=2}^{n+1} \binom{k}{2} &= \sum_{k=2}^{n+1} \frac{k(k-1)}{2} \\
&= \frac{1}{2}\left(\sum_{k=2}^{n+1} k^2 - \sum_{k=2}^{n+1} k\right) \\
&= \frac{1}{2}\left(\left(\sum_{k=1}^{n+1} k^2 - 1^2\right) - \left(\sum_{k=1}^{n+1} k - 1\right)\right) \\
&= \frac{1}{2}\left(\frac{(n+1)(n+2)(2n+3)}{6} - 1 - \frac{(n+1)(n+2)}{2} + 1\right) \\
&= \frac{1}{2}\left(\frac{2n(n+1)(n+2)}{6}\right) = \boxed{\frac{n(n+1)(n+2)}{6}}
\end{aligned}$$

b. *Base Case:* $n = 1$

$1 = \frac{1(2)(3)}{6}$, therefore the base case holds.

Induction Hypothesis: Assume $n = j$ holds

Assume $\sum_{k=2}^{j+1} \binom{k}{2} = \frac{j(j+1)(j+2)}{6}$ for some arbitrary integer j.

Induction Step: Prove $n = j + 1$ holds

$$\begin{aligned}
\sum_{k=2}^{j+2} \binom{k}{2} &= \sum_{k=2}^{j+1} \binom{k}{2} + \binom{j+2}{2} \\
&= \frac{j(j+1)(j+2)}{6} + \frac{3(j+2)(j+1)}{2 \cdot 3} \\
&= \boxed{\frac{(j+1)(j+2)(j+3)}{6}}
\end{aligned}$$

Therefore, by induction, this expression holds.