Problem Set 8: Stars and Bars, Pascal's Triangle, Combinatorial Proofs, Binomial Theorem

CS 198-087: Introduction to Mathematical Thinking
UC Berkeley EECS
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This homework is due on Wednesday, November 7th, at 11:59PM, on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX.

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1. *Fun with Stars and Bars*

   Determine the number of non-negative integer solutions to the equation

   $$x_1 + x_2 + x_3 + x_4 = 19$$

   subject to each of the following conditions:

   a. $x_1 \geq 0, x_2 > 4, x_3 \geq 2, x_4 \geq 1$

   b. $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, 0 \leq x_4 \leq 2$ *(Hint: Break this up into cases.)*

Solution:

a. Let's do a change of variables into some x_1', x_2', x_3' and x_4' such that the only conditions on x_i' are that $x_i' \geq 0$. Then, we can use the standard solution to stars and bars to determine the number of solutions.

   Notice that the condition $x_2 > 4$ is equivalent to $x_2 \geq 5$.

   $$\begin{aligned} x_1' &= x_1 \\ x_2' &= x_2 - 5 \\ x_3' &= x_3 - 2 \\ x_4' &= x_4 - 1 \end{aligned}$$

   Then, we have

$$x_1 + x_2 + x_3 + x_4 = 19$$
$$x_1 + (x_2 - 5) + (x_3 - 2) + (x_4 - 1) = 19 - 5 - 2 - 1$$
$$x_1' + x_2' + x_3' + x_4' = 11$$

Now, we can treat our problem as if we have 11 stars and 3 bars, yielding

$\boxed{\binom{11+3}{3} = \binom{14}{3}}$ integer solutions to this equation.

b. Notice the inequality on x_4 is now an upper bound, as opposed to a lower bound. This means there are three cases to consider: $x_4 = 0, x_4 = 1$ and $x_4 = 2$.

Case 1: $x_4 = 0$

We are now looking at the number of non-negative integer solutions to $x_1 + x_2 + x_3 = 19$, which is given by $\binom{21}{2}$.

Case 2: $x_4 = 1$

We are now looking at $x_1 + x_2 + x_3 + 1 = 19$, i.e. $x_1 + x_2 + x_3 = 18$, which has $\binom{20}{2}$ non-negative integer solutions.

Case 3: $x_4 = 2$

Following the pattern, there are $\binom{19}{2}$ non-negative integer solutions to $x_1 + x_2 + x_3 = 17$.

Our total number of solutions is then

$\boxed{\binom{19}{2} + \binom{20}{2} + \binom{21}{2} = 571}$.

2. *Combinatorial Proof* — n^2

In this problem, we'll investigate the identity

$$n^2 = \binom{n}{2} + \binom{n+1}{2}$$

a. Give an interpretation of this identity in terms of Pascal's Triangle.

b. Give a combinatorial proof of the identity $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n+1}{k+1}$.

c. Give a combinatorial proof of $n^2 = \binom{n}{2} + \binom{n+1}{2}$. You will need to use the identity above as an intermediate step.

Solution:

a. In lecture 8, we showed this identity: the squares of some number on the $\binom{n}{1}$ diagonal is equal to the sum of the number immediately to the right of it, $\binom{n}{2}$, and the number below the two, $\binom{n+1}{2}$.

b. Also done in lecture. This is Pascal's identity.

   LHS: Number of ways to choose $k+1$ people from a group of $n+1$.

   RHS: Suppose we want to choose $k+1$ people from a group of $n+1$. Suppose the people are numbered $p_1, p_2, ...p_{n+1}$. Consider the very first person: either we include them in our subset or do not include them.

   - If we include them, there are n people remaining and we need to choose k of them: $\binom{n}{k}$
   - If we do not include them, there are n people remaining and we need to choose $k+1$ of them: $\binom{n}{k+1}$

   Thus, the total number of ways to choose $k+1$ people from a group of $n+1$ is $\binom{n}{k} + \binom{n}{k+1}$. We've already shown this quantity is $\binom{n+1}{k+1}$, though, so these expressions both must be the same.

c. First, we use Pascal's identity to rewrite $\binom{n+1}{2}$ as $\binom{n}{1} + \binom{n}{2}$. Now, we need a combinatorial argument as to why $n^2 = \binom{n}{1} + 2\binom{n}{2}$.

   LHS: Number of possible outcomes when rolling two n-sided die.

   RHS: When rolling two n-sided die, either both dice rolls are the same, or they are different.

   - There are n potential outcomes for a single die, so there are $n = \binom{n}{1}$ outcomes where both dice rolls had the same output
   - If the rolls are different, there are $\binom{n}{2}$ ways to select two different numbers. However, with dice rolls, order matters, so we multiply by 2 to account for this (i.e. we want the sequence of rolls (A, B) to be different than the sequence of rolls (B, A)).

   This combination yields $\binom{n}{1} + 2\binom{n}{2}$, which is the RHS.

3. *Hockey Stick Theorem*

The "Hockey Stick Theorem" is shown on Pascal's Triangle below:

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

In the above example, $56 = 1+3+6+10+15+21$. Algebraically, the Hockey Stick Theorem states for any integers n, r such that $n \geq r$:

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}$$

a. Prove this statement using induction.

b. Prove this statement algebraically, using Pascal's Identity.

c. Prove this statement using a combinatorial proof.

Solution:

a. It's not immediately clear what the base case should be. We will do induction on n, and since we know that r is some value such that $n \geq r$, we will set our base case to be $n = r$.

Base Case: $n = r$

$$\sum_{k=r}^{r} \binom{k}{r} = \binom{r}{r} = 1, \quad \binom{n+1}{r+1} = \binom{r+1}{r+1} = 1$$

Therefore, the base case holds.

Induction Hypothesis: $n = j$ Assume $\sum_{k=r}^{j} \binom{k}{r} = \binom{j+1}{r+1}$ for some arbitrary $j \in \mathbb{N}, j \geq r$. We chose j as our arbitrary value instead of k because k already appears in the expression we're trying to prove.

Induction Step: We now need to show that the statement holds for $n = j+1$.

$$\begin{aligned}
\sum_{k=r}^{j+1} \binom{k}{r} &= \sum_{k=r}^{j} \binom{k}{r} + \binom{j+1}{r} \\
&= \binom{j+1}{r+1} + \binom{j+1}{r} \\
&= \binom{j+2}{r+1} \text{ \textit{(from Pascal's identity)}}
\end{aligned}$$

Therefore, by induction, the statement holds.

b. Let $a = n - r$. Repeatedly, we can combine two terms into one, using Pascal's identity.

$$\begin{aligned}\sum_{k=r}^{n} \binom{k}{r} &= \binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \ldots + \binom{r+a}{r} \\ &= \binom{r+1}{r+1} + \binom{r+1}{r} + \binom{r+2}{r} + \ldots + \binom{r+a}{r} \\ &= \binom{r+2}{r+1} + \binom{r+2}{r} + \binom{r+3}{r} + \ldots + \binom{r+a}{r} \\ &= \binom{r+3}{r+1} + \binom{r+3}{r} + \ldots + \binom{r+a}{r} \\ &\vdots \\ &= \binom{r+a}{r+1} + \binom{r+a}{r} \\ &= \binom{r+a+1}{r+1} = \boxed{\binom{n+1}{r+1}}\end{aligned}$$

c. Let's consider the number of ways we can distribute n pieces of candy to k children. This is a typical stars-and-bars setup, with n stars and $k-1$ bars, giving $\binom{n+k-1}{k-1}$ ways.

Alternatively, we could first fix the amount of candy we give child 1, which we can arbitrarily decide to be the oldest child. If we give them i pieces, where $0 \leq i \leq n$, then we have to distribute the remaining $n - i$ pieces amongst the remaining $k - 1$ children. In each of these subcases, there are $n - i$ stars and $k - 2$ bars. The sum of all of these subcases should be equal to $\binom{n+k-1}{k-1}$ as we saw in the direct case, so we have the following relation:

$$\sum_{i=0}^{n} \binom{n-i+k-2}{k-2} = \binom{n+k-1}{k-1}$$

This looks similar to the Hockey Stick theorem, but not quite. We can fit it to the form we want by using the substitutions $m = n + k - 2$ and $r = k - 2$ (using the fact that $r = m - n$):

$$\binom{m+1}{r+1} = \sum_{i=0}^{n} \binom{m-i}{r} = \sum_{i=r}^{m} \binom{i}{r}$$

Going from $i = 0$ to $i = n$, the values of $\binom{m-i}{r}$ are $\binom{m}{r}$, $\binom{m-1}{r}$, ..., $\binom{r+1}{r}$, $\binom{r}{r}$ (where we used the substitution $m - n = r$). This is equivalent to summing $\binom{i}{r}$ from $i = r$ to $i = m$, as that yields the sequence $\binom{r}{r}$, $\binom{r+1}{r}$, ..., $\binom{m-1}{r}$, $\binom{m}{r}$.

Credits to AoPS, which I heavily relied on while creating this problem and its solution.

4. *Optional — More Practice with Combinatorial Proofs*

   Give combinatorial proofs for each of the following statements.

   a. $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$, where $n \geq r \geq k$

   b. $\binom{2n}{n} = 2\binom{2n-1}{n-1}$ *(Hint: Notice,* $\binom{2n-1}{n-1} = \binom{2n-1}{n}$*)*

Solution:

a. Suppose I have n basketball players, from which I want to select a team of r. Furthermore, k of my r players will be considered "captains."

   LHS: I could first select all of my r players, which I can do in $\binom{n}{r}$ ways, and then select my k captains from those r, which I can do in $\binom{r}{k}$ ways, giving a total of $\binom{n}{r}\binom{r}{k}$.

   RHS: I could also first select my k captains, which I can do in $\binom{n}{k}$ ways, and then select my remaining $r - k$ players from the general pool, which I can do in $\binom{n-k}{r-k}$ ways, giving a total of $\binom{n}{k}\binom{n-k}{r-k}$ ways.

   Therefore, $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$.

b. Using the hint, we rewrite our equation as $\binom{2n}{n} = \binom{2n-1}{n} + \binom{2n}{n-1}$. This is then just a restatement of Pascal's identity.

5. *Binomial Theorem — General Term*

   Let $g(x) = (2x^5 - 3x^2)^7$.

   a. What is the sum of the coefficients of the expansion of $g(x)$?

   b. Find the general term of the expansion of $g(x)$.

   c. What is the coefficient on x^{20}?

   d. What is the coefficient on x^{18}?

Solution:

a. To find the sum of the coefficients in an expansion, we set the values of all variables to 1. In our case, $g(1) = (2 \cdot 1^5 - 3 \cdot 1^2)^7 = (2 - 3)^7 = (-1)^7 = -1$, meaning the sum of the coefficients of this expansion is $\boxed{-1}$.

b.

$$\begin{aligned} t_k &= \binom{7}{k}(2x^5)^{7-k}(-3x^2)^k \\ &= \boxed{(-1)^k\binom{7}{k}2^{7-k}3^k x^{35-3k}} \end{aligned}$$

as required.

c. To find the coefficient on x^{20}, we set the exponent $35-3k=20$, and solve to find $k=5$. We then substitute $k=5$ into the general term, yielding $t_5=(-1)^5\binom{7}{5}2^{7-5}3^5x^{20}$, meaning the coefficient on x^{20} is $\boxed{(-1)^5\binom{7}{5}2^{7-5}3^5=-20412}$

d. Solving $35-3k=18$ yields $k=\frac{17}{3}$. Since k is our index for term number in the binomial expansion, there is no meaning for non-integer values of k. This means x^{18} does not appear in the expansion of $g(x)$, meaning the coefficient is $\boxed{0}$.

6. *Approximations with the Binomial Theorem*

Use the first three terms of the Binomial Theorem to approximate the following values:

The combinatorial term $\binom{n}{k}$ is only valid when n and k are integers, since factorials (as we've seen them so far) are only defined for integers.

However, we can rewrite terms of the form $\binom{n}{k}$ so that they don't involve factorials, e.g.

$$\binom{n}{2}=\frac{n!}{2!(n-2)!}=\frac{n(n-1)}{2}$$
$$\binom{n}{3}=\frac{n!}{3!(n-3)!}=\frac{n(n-1)(n-2)}{6}$$

This allows us to approximate numbers raised to non-integer powers. For example, suppose we want to approximate $8.03^{\frac{1}{3}}$. Using just the first few terms of the binomial expansion yield:

$$\begin{aligned} 8.03^n &= (8+0.03)^n \\ &= 8^n + n\cdot 8^{n-1}\cdot 0.03 + \frac{n(n-1)}{2}8^{n-2}\cdot 0.03^2 \\ &= 8^{\frac{1}{3}} + \frac{1}{3}\cdot 8^{-\frac{2}{3}}\cdot 0.03 + \frac{\frac{1}{3}\cdot(-\frac{2}{3})}{2}8^{-\frac{5}{3}}\cdot 0.03^2 \\ &= 2.002496875 \end{aligned}$$

With a calculator, we have that this expansion is equal to 2.002496875, which is very close to the true value of 2.00249688.

Use the first three terms of the binomial expansion to approximate each of the following values. Use a calculator if need be, but only when absolutely necessary.

a. 5.02^3

b. $9.08^{\frac{1}{2}}$

c. $31^{-\frac{1}{5}}$

Solution:

a. In part(a), our exponent is an integer, so the preface to this problem really doesn't apply. We can proceed as normal.

$$\begin{aligned}
5.02^3 &= (5+0.02)^3 \\
&= 5^3 + \binom{3}{1}5^2 \cdot 0.02 + \binom{3}{2}5^1 \cdot 0.02^2 + \binom{3}{3}0.02^3 \\
&= 125 + 75 \cdot 0.02 + 15 \cdot 0.0004 + 0.000008 \\
&= \boxed{126.506008}
\end{aligned}$$

Here, this value is exact, because we completed the binomial expansion.

b. We'll use the fact that we know that $9^{\frac{1}{2}} = 3$.

$$\begin{aligned}
9.08^{\frac{1}{2}} &= (9+0.08)^{\frac{1}{2}} \\
&= 9^{\frac{1}{2}} + (\frac{1}{2})9^{-\frac{1}{2}}0.08 + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}9^{-\frac{3}{2}}0.08^2 \\
&= 3 + \frac{1}{2} \cdot \frac{1}{3} \cdot 0.08 - \frac{1}{8} \cdot \frac{1}{27} \cdot 0.08^2 \\
&= \boxed{3.0133037}
\end{aligned}$$

Using a calculator gives $9.08^{1/2} = 3.0133038$, which is very close to our result from using just 3 binomial expansion terms.

Note: This expansion is infinite. For integer exponents, we follow the sequence $\binom{n}{0}$, $\binom{n}{1}$, ... which ends at $\binom{n}{n}$. However, for fractions, the sequence 1, n, $\frac{n(n-1)}{2}$, $\frac{n(n-1)(n-2)}{6}$, $\frac{n(n-1)(n-2)(n-3)}{24}$, ... has no end (as no term $n-i$ will ever be equal to 1).

Additionally, if this sequence had some end, it would imply $9.08^{1/2}$ is rational (which it is not).

c. Now, we have a negative exponent. This doesn't change our process, though! We use the fact that $32^{-\frac{1}{5}} = \frac{1}{2}$. Then:

$$
\begin{aligned}
31^{-\frac{1}{5}} &= (32-1)^{-\frac{1}{5}} \\
&= 32^{-\frac{1}{5}} + (-\frac{1}{5}) \cdot 32^{-\frac{6}{5}}(-1) + (-\frac{1}{5})(-\frac{6}{5})32^{-\frac{11}{5}}(-1)^2 \\
&= \frac{1}{2} + (\frac{1}{5})\frac{1}{2^6} - \frac{6}{25} \cdot \frac{1}{2^{11}} \\
&= \boxed{0.5030078125}
\end{aligned}
$$

Here, our solution isn't as accurate with just 3 terms, as a calculator tells us $31^{-\frac{1}{5}} = 0.503184971$. However, we did identify the value correctly to the first three decimal places, and with more terms we would converge on the solution.