PROBLEM SET 9: BINOMIAL THEOREM, VIETA'S FORMULAS

CS 198-087: INTRODUCTION TO MATHEMATICAL THINKING
UC BERKELEY EECS
FALL 2018

This homework is due on Wednesday, November 14th, at 11:59PM, on Gradescope. As usual, this homework is graded on participation, but it is in your best interest to put full effort into it. This is a good opportunity to learn how to use LaTeX.

---

1. *Freshman's Dream*

   In modular arithmetic, the "freshman's dream" identity is as follows:

   $$(x+y)^p \equiv x^p + y^p \pmod{p}$$

   for prime p. Prove this identity.

   (Hint: You will need to use the binomial theorem. First, prove that $\binom{p}{i} \equiv 0 \pmod{p}$ for all $i \in \{1, 2, 3, ...p-1\}$.)

2. *Determining Coefficients*

   a. Determine the coefficient of x^{50} in the expansion of

   $$(x+1)^{1000} + x(x+1)^{999} + x^2(x+1)^{998} + ... + x^{999}(x+1) + x^{1000}$$

   (Hint: You may need to use the Hockey Stick identity.)

   b. Determine the coefficient of x^3 in the expansion of

   $$(x^2 + x - 5)^3$$

3. *Evaluating Sums*

   Evaluate the sum

   $$\sum_{k=0}^{n} k \binom{n}{k} (-1)^{k-1} 3^{n-k}$$

   (Hint: Replace -1 with a variable. What is this sum the derivative of?)

4. *Product of Multiple Binomial Expansions*

   Let's explore another application of the binomial theorem. Let $f(x,y) = (2x - 3y)^5$ and $g(x,y) = (x^3 - 3xy^2)^9$.

   a. Find the general terms of both $f(x,y)$ and $g(x,y)$. Use the index variable k for $f(x,y)$ and i for $g(x,y)$.

   b. Find the combined general term, that is, find the general term of $f(x,y) \cdot g(x,y)$. It will be of the form $t_{k,i} = \binom{5}{k}\binom{9}{i}\ldots$

   c. Find the sum of the coefficients of the product $f(x,y) \cdot g(x,y)$.

   d. Determine all terms containing x^{14} in the expansion of $f(x,y) \cdot g(x,y)$.

5. *Arguing about Complex Roots with Vietas*

   Suppose $p(x) = a_2x^2 + a_1x + a_0$, with $a, b, c \in \mathbb{Z}$, has roots r_1 and r_2, where r_1, r_2 are potentially complex.

   a. Prove, using Vieta's formulas, that r_1 is real if and only if r_2 is real.

   b. Prove, using Vieta's formulas, that if $r_1 = a + bi$, then r_2 is the complex conjugate of r_1, i.e. $r_2 = a - bi$. *(Hint: Start by assuming r_1 is an arbitrary complex number $c + di$, and that the only values that work for c, d are $c = a$ and $d = -b$.)*

6. *Determining Roots*

   Suppose $f(x) = x^3 - 3x^2 + 1$ has roots a, b, c.

   a. Find a polynomial that has roots $a + 3, b + 3, c + 3$. *(How do we shift $f(x)$ three units to the right?)*

   b. Find a polynomial that has roots $\frac{1}{a+3}, \frac{1}{b+3}, \frac{1}{c+3}$.

   c. Determine $\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3}$.

   d. Find a polynomial that has roots a^2, b^2, c^2.

7. *Sums of Coefficients*

   a. Three roots of $x^4 + ax^2 + bx + c = 0$ are $9, -3$ and 2. Determine $a + b + c$.

   b. If $P(x)$ is a polynomial such that

   $$x^{23} + 23x^{17} - 18x^{16} - 24x^{15} + 108x^{14} = (x^4 - 3x^2 - 2x + 9)P(x)$$

   determine the sum of the coefficients of $P(x)$.

8. *Reciprocal Polynomials*

   Given some polynomial $p(x)$ of degree n, we define its reciprocal polynomial $p^*(x)$ as

   $$p^*(x) = x^n p\left(\frac{1}{x}\right)$$

   There are a few special properties of $p^*(x)$; let's demonstrate with an example.

Suppose $p(x) = ax^2 + bx + c$ has roots r_1, r_2.

$$\begin{aligned} p^*(x) &= x^2 p\left(\frac{1}{x}\right) \\ &= x^2\left(a \cdot \frac{1}{x^2} + b \cdot \frac{1}{x} + c\right) \\ &= cx^2 + bx + a \end{aligned}$$

We notice that the coefficients of $p^*(x)$ (c, b, a) are the reverse of the coefficients of $p(x)$ (a, b, c).

Now, suppose we want to find some polynomial that has roots $\frac{1}{r_1}$ and $\frac{1}{r_2}$. We proceed as follows:

$$\begin{aligned} 0 = \left(x - \frac{1}{r_1}\right)\left(x - \frac{1}{r_2}\right) &= x^2 - \frac{r_1 + r_2}{r_1 r_2}x + \frac{1}{r_1 r_2} \\ &= r_1 r_2 x^2 - (r_1 + r_2)x + 1 \\ &= \frac{c}{a}x^2 + \frac{b}{a}x + 1 \\ &= cx^2 + bx + a \\ &= p^*(x) \end{aligned}$$

It turns out that this polynomial is precisely $p^*(x)$!

There are two main takeaways here:

a. The coefficients of $p^*(x)$ are the reverse of the coefficients of $p(x)$

b. If $p(x)$ has roots $r_1, r_2, ..., r_n$, then $p^*(x)$ has roots $\frac{1}{r_1}, \frac{1}{r_2}, ..., \frac{1}{r_n}$

Prove both statement (a) and statement (b) for polynomials of arbitrary degree.