Lecture 10: Series and Sequences

http://book.imt-decal.org, Ch. 2.3

Introduction to Mathematical Thinking

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Announcements

- Homework 4 due tomorrow, Gradescope
  - No late submissions: I like to post solutions on Saturday.
- Might be a little delay in getting Homework 5 out (until Sunday), but it will be relatively short.
- Next class: starting modular arithmetic and number theory.
  - Textbook section will be written by Tuesday (doesn't exist yet).

Series and Sequences

Now, we'll look at formulas for the sums of arithmetic sequences, as well as sums of the form $\sum_{i=1}^{n} i^k$.

Note: You may have seen some of these sequences in CS 61A/B, when learning about runtime analysis.

Arithmetic Sequences

An arithmetic sequence is defined as

common diff

$$t_1 = a \quad t_k = t_{k-1} + d, k \in \mathbb{N}$$

where $d \in \mathbb{R}$. We can also express a general term of an arithmetic sequence without recursion:

$$t_k = a + (k-1)d$$

$t_1 = a$

$t_2 = a + d$

$t_3 = a + 2d$

$\vdots$

For example,

$$\underbrace{3, 10}_{+7}, \underbrace{10, 17}_{+7}, \ldots$$

$$3, 10, 17, 24, 31, 38, \ldots$$

+7 +7 +7 +7 +7

is an arithmetic sequence with $a = 3$ and $d = 7$.

initial term

Now: Suppose we want to determine the sum of the first n terms of an arithmetic sequence, i.e. $\sum_{k=1}^{n}(a + (k-1)d)$.

$1, 2, 3, 4, \ldots$

$a = 1$

$d = 1$

Sum of First n Natural Numbers (Arithmetic Series)

Before determining the sum of an arbitrary arithmetic sequence, let's start with the most basic arithmetic sequence ~~Ã¢â‚¬â€œ~~ 1, 2, 3, 4, ... Specifically, we want to find an expression for $\sum_{i=1}^{n} i$.

$$
\begin{aligned}
& S_n = 1 + 2 + 3 + \cdots + n \\
+ \quad & S_n = n + (n-1) + (n-2) + \cdots + 1 \\
\hline
& 2S_n = n(n+1)
\end{aligned}
$$

$$\Rightarrow \quad S_n = \frac{n(n+1)}{2} = \sum_{i=1}^{n} i$$

Direct Proof

Derivation

General Arithmetic Series

$$\sum_{i=1}^{n} \left(a + (i-1)d\right)$$

series: sum of sequence

$$S_n = a + (a+d) + (a+2d) + \cdots + (a+(n-2)d) + (a+(n-1)d)$$

$$S_n = (a+(n-1)d) + (a+(n-2)d) + (a+(n-3)d) + \cdots + (a+d) + a$$

+

$$2S_n = (2a+(n-1)d) + (2a+(n-1)d) + \cdots + (2a+(n-1)d)$$

$$2S_n = n\left(2a+(n-1)d\right)$$

$$\Rightarrow S_n = \frac{n\left(2a+(n-1)d\right)}{2} = \frac{\text{first} + \text{last}}{2} \cdot (\#\text{ of terms})$$

$$= (\text{average}) \cdot (\#\text{ terms})$$

Proof of Arithmetic Series Formula, using Induction

RTP: $\sum_{i=1}^{n}(a+(i-1)d) = \frac{2a+(n-1)d}{2}n$

Base Case: $n = 1$

LHS: $\sum_{i=1}^{1}(a+(i-1)d) = a + 0d = a$

RHS: $\frac{2a+0d}{2} = a$

LHS = RHS, therefore the base case holds.

Induction Hypothesis: Assume that $\sum_{i=1}^{k}(a+(i-1)d) = \frac{2a+(k-1)d}{2}k$, for some arbitrary k.

Induction Step:

$$\sum_{i=1}^{k+1}(a+(i-1)d) = \sum_{i=1}^{k}(a+(i-1)d) + a + kd$$

$$= \frac{2a+(k-1)d}{2}k + \frac{2a+2kd}{2}$$

Therefore, induction holds.

Geometric Sequences

A **geometric sequence** is a sequence of numbers defined by a starting term a and a common ratio r. In the sequence, the ratio of consecutive terms (i.e. $\frac{t_n}{t_{n-1}}$) is constant, and is equal to r.

162

$$2, 6, 18, 54, \cancel{108}, \ldots$$

$\times 3 \quad \times 3 \quad \times 3$

$t_1 = 2$

$t_2 = 2 \cdot 3$

$t_3 = 2 \cdot 3^2$

$t_4 = 2 \cdot 3^3$

$\vdots$

In general, we have that

$$t_k = ar^{k-1}$$

represents the kth term in the sequence, assuming that we start counting at 1. As with arithmetic sequences, we can also phrase a geometric sequence recursively:

$$t_1 = a$$

$$t_k = rt_{k-1}, k \geq 1$$

Geometric Series

$$\sum_{i=1}^{n} ar^{i-1}$$

Now, we want to find an expression for $\cancel{\sum_{i=1}^{n} ar^{n-1}}$.

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \dots + \cancel{ar^{n-2}} + \cancel{ar^{n-1}}$$

$$rS_n = \cancel{ar} + \cancel{ar^2} + \cancel{ar^3} + \dots + \cancel{ar^{n-1}} + ar^n$$

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$\Rightarrow \quad S_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{a(r^n-1)}{r-1}$$

$$r^n - 1 = (r-1)(r^{n-1} + r^{n-2} + \dots + r^2 + r + 1)$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

e.g. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

Assume $|r| < 1$

$r^n \to 0$
when $|r| < 1$

$$\lim_{n \to \infty} \frac{a(1-r^n)}{1-r} = \boxed{\frac{a}{1-r}}$$

sum of an infinite geometric series

Telescoping Sums

$a_i = i$

Consider the sum

$$\sum_{i=1}^{99}(a_{i+1} - a_i) = \underbrace{(2-1)}_{1} + \underbrace{(3-2)}_{1} + \underbrace{(4-3)}_{1} + \underbrace{(5-4)}_{1} + \dots + \underbrace{(100-99)}_{1}$$

Is there any way we can rearrange the terms in this sum in order to make the calculation easiers?

$$= -1 + (2-2) + (3-3) + (4-4) + \dots + (99-99) + 100$$

$$= -1 + 100 = 99$$

Example

Let's evaluate $\sum_{k=1}^{100}(\cos(k) - \cos(k-1))$.

$$\sum_{k=1}^{100}\big(\cos(k) - \cos(k-1)\big) = (\cos 1 - \cos 0) + (\cos 2 - \cos 1) + \dots + (\cos 100 - \cos 99)$$

$$= -\cos 0 + (\cos 1 - \cos 1) + (\cos 2 - \cos 2) + \dots + (\cos 99 - \cos 99) + \cos 100$$

$$= \cos 100 - \cos 0$$

Example

$$\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \ldots$$

Evaluate

$$\sum_{i=1}^{n} \frac{1}{i(i+1)}$$

$$\frac{1}{i} - \frac{1}{i+1}$$

$$= \frac{(i+1)}{i(i+1)} - \frac{i}{i(i+1)} = \frac{1}{i(i+1)}$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 + \left(\frac{1}{2} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n-1}\right) + \left(\frac{1}{n} - \frac{1}{n}\right) - \frac{1}{n+1}$$

$$= \boxed{1 - \frac{1}{n+1}}$$

Sums of Powers of First n Natural Numbers

We want to use the power of telescoping sums to derive identites for sums of the form $\sum_{i=1}^{n} i^k$, i.e. $1+2+3+...+n$ or $1^2+2^2+...+n^2$.

Key insight: Consider the expansion of $(i+1)^2$:

$$(i+1)^2 = i^2 + 2i + 1$$

How can we leverage this to determine a sum for $\sum_{i=1}^{n} i$?

$$(i+1)^2 = i^2 + 2i + 1$$

$$\Rightarrow \underbrace{(i+1)^2 - i^2}_{} = 2i+1$$

$$\sum_{i=1}^{n} c x_i$$

$$= c \sum_{i=1}^{n} x_i$$

$5+10+15 = 5(1+2+3)$

Notice, if we take the sum on both sides of the equation, the equality will still hold.

$$\Rightarrow \sum_{i=1}^{n}[(i+1)^2 - i^2] = \sum_{i=1}^{n}(2i+1)$$

$$(2^2-1^2)+(3^2-2^2)+\ldots+(n^2-(n-1)^2)+((n+1)^2-n^2) = \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 1$$

$$-1^2+(2^2-2^2)+(3^2-3^2)+\ldots+(n^2-n^2)+(n+1)^2 = 2\sum_{i=1}^{n} i + n$$

S

$$(n+1)^2 - 1 = 2S + n$$

$$2S = (n+1)^2 - n - 1 = n^2 + 2n + \cancel{1} - n - \cancel{1}$$

$$\Rightarrow 2S = n^2 + n \Rightarrow S = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$$

$1+2+\dots+n : (i+1)^2$

$1^2+2^2+\dots+n^2 : (i+1)^3$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(i+1)^3 = i^3+3i^2+3i+1$$

$$\sum_{i=1}^{n}\left((i+1)^3-i^3\right) = \sum_{i=1}^{n}\left(3i^2+3i+1\right)$$

$$(n+1)^3-1 = 3\underbrace{\boxed{\sum_{i=1}^{n} i^2}}_{S} + 3\underbrace{\sum_{i=1}^{n} i}_{2} + \sum_{i=1}^{n} 1$$

$$(n+1)^3-1 = 3S + 3\frac{n(n+1)}{2} + n$$

$$3S = (n+1)^3-1-3\frac{(n)(n+1)}{2}-n$$

$$3S = \frac{n^3+3n^2+3n+1-1-3n^2-3n+n}{2} = \frac{n(n+1)(2n+1)}{2}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \dots + n^3$$

$\rightarrow$ look at $(i+1)^4$.