Lecture 17: Counting

Introduction to Mathematical Thinking

April 2, 2019

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Announcements

- I'm glad to be back!
- Homework 6 due on Sunday
- **Quiz 4 on Tuesday, in class** (a week from today)
    - Will cover everything since the last quiz (counting) through Thursday
- The textbook is now formatted as a set of notes, at notes.imt-decal.org
    - Notes are linked in the table wherever relevant
    - "Key Examples in Counting" and "Stars and Bars" are especially relevant for the homework and quiz

Current Plan: Finish up counting today and Thursday

- Rest of Thursday – Guerilla-section style walkthrough of Homework 6 problems
- **Won't review everything that was covered by Ani – refer to his slides**

Sum and Product Rules

Product rule: Suppose we have to make a series of k choices. If we have n_1 choices at step 1, n_2 choices at step 2, ..., n_k choices at step k, where each choice is independent of one another: total number of choices is $n_1 \cdot n_2 \cdot ... \cdot n_k$

- For example: How many factors does 48 have?

1, 2, 3, 4, 6, 8, 12, 16, ... too long / too lazy

$48 = 2^4 \cdot 3$

$2^a 3^b$

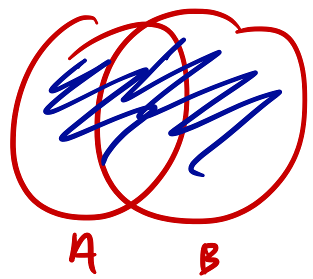
$0 \le a \le 4 \quad 0 \le b \le 1$

$a = 0, 1, 2, 3, 4 \quad b = 0, 1$

$\#\text{factors}(48) = 5 \cdot 2 = 10$

Sum rule: Suppose there are k **disjoint** (non-overlapping) possibilities/cases, of size $n_1, n_2, ..., n_k$. Then, the total number of possibilities is $n_1 + n_2 + ... + n_k$.

Principle of Inclusion-Exclusion



We will now use PIE to help us solve counting-style questions. Recall, for sets A, B:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Suppose there are 50 students at Billy High. Each student is enrolled in at least one of the two math classes the course offers. 40 students are enrolled in calculus, and 25 are enrolled in linear algebra. How many students are in both?

Let x be the number of students in both. Then, from PIE:

$$50 = 40 + 25 - x$$

$$\Rightarrow x = 15$$

Therefore, there are 15 students enrolled in both.

Example: How many numbers integers between 1 and 1000 (inclusive) are multiples of 3 or 5?

$3: \quad 3\cdot 1,\ 3\cdot 2,\ \ldots \quad 3\cdot 333 \quad \boxed{333}$

$5: \quad 5\cdot 1,\ 5\cdot 2,\ \ldots \quad 5\cdot 200 \quad \boxed{200}$

$15: \quad 15\cdot 1,\ 15\cdot 2,\ \ldots \quad 15\cdot 66 \quad \boxed{66}$

$= 990$

$\text{Total} = 333 + 200 - 66 = \boxed{467}$

$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$

Permutations

The number of ways to arrange k items, selected from a group of n, where **order matters** is

$$P(n, k) = \frac{n!}{(n-k)!}$$

For example: suppose I have 5 people, named Alpha, Beta, Charlie, David and Edgar.

- How many ways can I arrange them in a line?

$\underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1} = 5!$

- How many ways can I arrange 3 of them in a line?

$\underline{5}\ \underline{4}\ \underline{3} = \frac{5!}{2!} = \frac{5!}{(5-3)!}$

Repeated Characters and Arrangements

How many permutations of the string "DAD" are there?

Is it $3! = 6$?

DAD, ADD, DDA

No: there are only 3. We need to account for the repeated D character.

$D_1 A D_2$

$D_1 A D_2$

$D_2 A D_1$

$A D_1 D_2$

$A D_2 D_1$

$D_1 D_2 A$

$D_2 D_1 A$

$$\frac{3!}{2!}$$

divide by # of ways to arrange repeated chars.

perms of "DOG"

$\underline{3}\ \underline{2}\ \underline{1}$

$= 3!$

Example: How many permutations are there of MISSISSIPPI? *(Hint: First, determine the number of each letter.)*

M 1

IIII 4

SSSS 4

PP 2

(# chars)!

$$\frac{11!}{4!\,4!\,2!}$$

repeats of I

repeats of S

repeats of P

Example: How many times does the substring "DOG" appear in all permutations of "BABYDOG"?

BB

A

y

$\boxed{DOG}$

treat DOG as a single character

$$\frac{5!}{2!}$$

A y BB DOG

" DGO

" OGD

" ODG

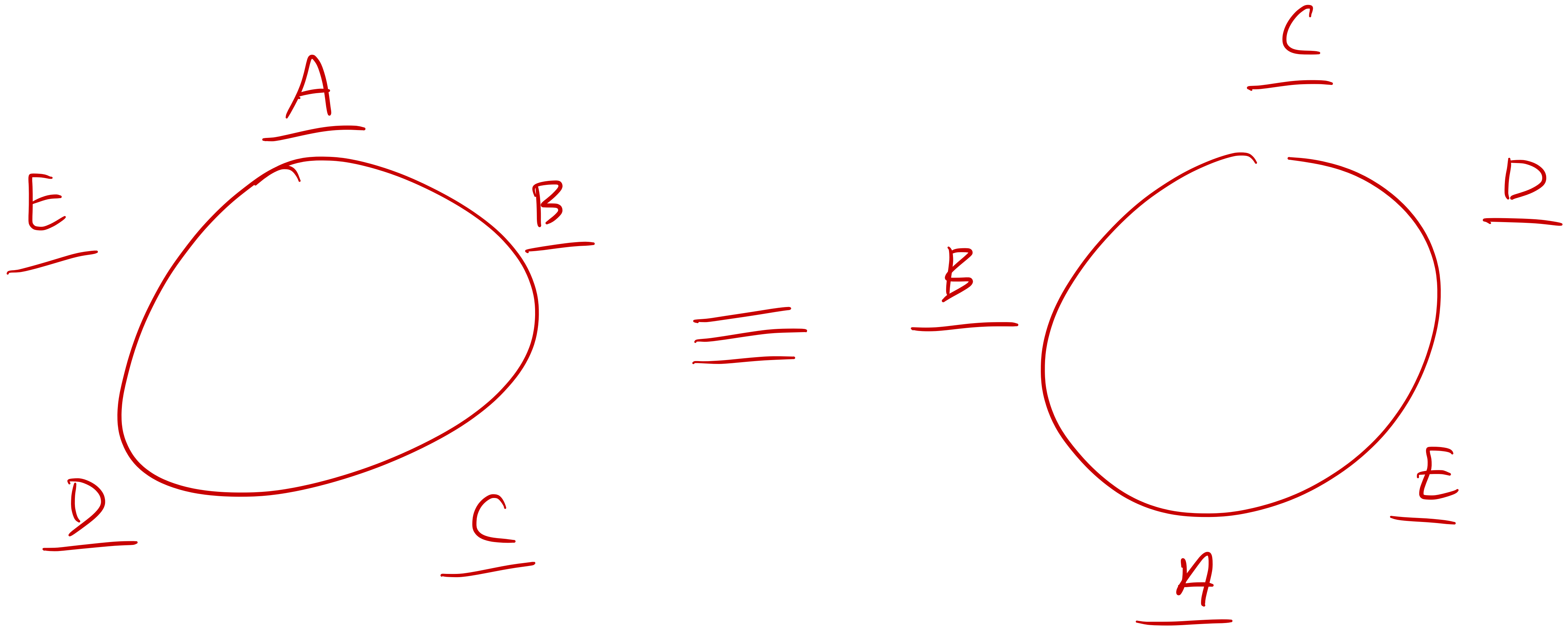
" GOD

" GDO

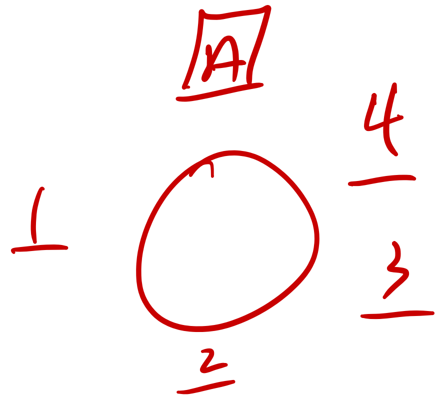
Followup: How many times do D, O, G appear together, not necessarily in that order?

$$\frac{5!}{2!} \cdot 3!$$

Example: How many ways can I seat 5 people around a circular dinner table?



$$\frac{5!}{5} = 4!$$



$= 4!$

Combinations

$$\underline{5}\ \underline{4}\ \underline{3} = \frac{5!}{2!}$$

Before, when selecting 3 of ABCDE, we said "ABE" and "BEA" were different selections (we used the analogy of arranging them in a line). There were $\frac{5!}{2!}$ such arrangements. In permutations, order matters.

Now, if we were to select 3 of ABCDE, where order does not matter, we would need to divide by the number of repeated arrangements that are really the same. In the above example,

ABE, AEB, BAE, BEA, EAB, EBA

are all the same. There are $3!$ repeats of the same arrangement, thus we must divide $\frac{5!}{2!}$ by $3!$ to find the number of arrangements without repeats, giving us $\frac{5!}{2!3!}$.

$$\frac{5!}{2!3!} = 10$$

In general, the number of ways to select k items from a group of n, where **order does not matter** is

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

This is read "n choose k", and is sometimes referred to as the binomial coefficient.

Note: $\binom{n}{k} = \binom{n}{n-k}$. This is because choosing k items to include is the same as choosing $n-k$ items to discard. In the previous example, instead of selecting 3 of ABCDE to include in our arrangement, we could've equivalently selected 2 to exclude.

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{k} k! = P(n, k)$$

Example: Suppose I want to form a team of 3 basketball players from a pool of 10.

a) How many teams can I form if there are no positions?

"modern NBA"

$$\binom{10}{3}$$

LB, KB, MJ

b) How many teams can I form, if there must be a guard, forward, and center?

$$\frac{10}{\text{guard}} \quad \frac{9}{\text{forward}} \quad \frac{8}{\text{center}}$$

$$= \binom{10}{3} 3!$$

guard: LB
forward: KB
center: MJ

different!

guard: MJ
forward: LB
center: KB

Total : $\binom{17}{8}$

Example: Suppose I want to select 8 children from a group of 8 boys and 9 girls. How many ways can I select 3 boys and 5 girls to form this group?

Number of ways to select boys: $\binom{8}{3}$
Number of ways to select girls: $\binom{9}{5}$

Total number of ways, using the product rule:

$$\binom{8}{3}\binom{9}{5}$$

Followup: How many ways can I select at least 4 boys to form this group?

1) 4 or 5 or 6 or 7 or 8

2) complementary counting : Total $-$ (0 or 1 or 2 or 3)

Direct

Choosing at least 4 boys is the same as choosing 4 boys OR 5 boys OR 6 boys OR 7 boys OR 8 boys. In each case, when we choose k boys, we need to also choose $8-k$ girls.

$$ways(\text{at least 4 b}) = ways(\text{4 b}) + ways(\text{5 b}) + ways(\text{6 b}) + ways(\text{7 b}) + ways(\text{8 b})$$

$$= \binom{8}{4}\binom{9}{4} + \binom{8}{5}\binom{9}{3} + \binom{8}{6}\binom{9}{2} + \binom{8}{7}\binom{9}{1} + \binom{8}{8}\binom{9}{0}$$

Complementary

$$= \binom{17}{8} - \binom{8}{0}\binom{9}{8} - \binom{8}{1}\binom{9}{7} - \binom{8}{2}\binom{9}{6} - \binom{8}{3}\binom{9}{5}$$

Example: Deck of cards

Recall, in a standard deck of cards, there are 52 cards. Each card has 1 of 4 suits (Spades, Clubs, Hearts, Diamonds) and 1 of 13 values (Ace, 2, 3, ... 10, Jack, Queen, King). In a hand of cards, the order does not matter.

1. How many 5 card hands are there in Poker?

$$\binom{52}{5}$$

$$\binom{12}{3} = \frac{12 \cdot 11 \cdot 10 \cdot \cancel{9!}}{\cancel{9!}\ 3!}$$

$$\binom{12}{3} 4^3 = \frac{48 \cdot 44 \cdot 40}{3!}$$

2. How many 5 card hands are there that include exactly one pair (values aabcd, e.g. 2 3s, or 2 5s, etc.)?

$$\underbrace{\binom{13}{1}}_{\text{face val of a}} \underbrace{\binom{4}{2}}_{\text{suits of a}} \underbrace{\binom{12}{3}}_{\text{fv of b, c, d}} 4^3 \leftarrow \text{suits of b, c, d}$$

$$\frac{48 \cdot 44 \cdot 40}{3!}$$

3. How many 5 card hands are there that include a four-of-a-kind (values aaaab, e.g. 4 3s and a 5)

$$\underbrace{\binom{13}{1}}_{\text{choosing face val of a}} \cdot \binom{12}{1} \cdot 4 = 13 \cdot 48$$

A − 5
2 − 6
3 − 7
4 − 8
5 − 9
6 − 10
7 − J
8 − Q
9 − K

4. How many 5 card hands are there that have a straight, i.e. where all card values are consecutive? (e.g. 3, 4, 5, 6, 7, but the suits don't matter)

$$9 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 9 \cdot 4^5$$

5. How many 5 card hands are there that are a straight flush, i.e. where all card values are consecutive and all cards are of the same suit? (e.g. 3, 4, 5, 6, 7, where all cards are diamonds)

$$\underbrace{9}_{\text{\# of straights}} \cdot 4 = 36$$

6. How many 5 card hands are there where all cards are of the same suit?

$$\underbrace{\binom{4}{1}}_{\text{suit}} \underbrace{\binom{13}{5}}_{\text{face values}}$$

Stars and Bars

How many ways can I distribute 12 pieces of candy to 3 of my friends?

- My three friends are **distinguishable**: they are all visibly different
- We will assume the pieces of candy are all **indistinguishable**: they are all the same

We will model this sort of situation with "stars and bars".

- 12 "stars", or items
- 3 - 1 = 2 "bars", or separators

$$\overset{2}{\underset{f_1}{**}} \mid \overset{6}{\underset{f_2}{******}} \mid \overset{4}{\underset{f_3}{****}}$$

This setup represents friend 1 getting 2 pieces, friend 2 getting 6 pieces and friend 3 getting 4 pieces.

$$\overset{f_1}{*} \mid \overset{f_2}{} \mid \overset{f_3}{***********}$$

This setup represents friend 1 getting 1 piece, friend 2 getting 0 pieces and friend 3 getting 11.

This problem boils down to finding the number of permutations of

12 stars
2 bars

$$* * * * * * * * * * * * ||$$

In our case, this is $\frac{14!}{12!2!} = \binom{14}{2} = \binom{14}{12}$

In general, the number of ways to arrange n **indistinguishable (i.e. identical) items** into k **distinguishable bins** is

$$\binom{n+k-1}{k-1}$$

Since k bins corresponds to $k-1$ bars, we can also write this number as

bars = bins - 1

$$\binom{\text{stars}+\text{bars}}{\text{stars}} = \binom{\text{stars}+\text{bars}}{\text{bars}}$$

Example: Determine the number of non-negative integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 25$$

$x_i \geq 0$

This fits the same model. We have 25 "stars", and 4 "bars", meaning there exist

$$\binom{29}{4}$$

solutions to this equation.

distributing 25 candies to 5 friends

$$\binom{25+4}{4} = \binom{25+4}{25}$$

Followup: How many *positive* integer solutions exist to this equation?

Previously, we had that $x_1, x_2, x_3, x_4, x_5 \geq 0$. We now want $x_i > 0$, or equivalently, $x_i \geq 1$.

To account for this, we can define $x_i' = x_i - 1$. Now, the only constraint we have is $x_i' \geq 0$, which we already know how to solve (from the last slide).

$x_i \geq 1$

$x_i - 1 \geq 0$

$\boxed{x_i' \geq 0}$

now use setup from prev. slide

The number of positive integer solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 25 \quad -5 \quad -5$$

is the same as the number of non-negative integer solutions to

$$x_1' + x_2' + x_3' + x_4' + x_5' = 25 - 5 = 20$$

which is $\binom{24}{4}$.

$\binom{20+4}{4}$

In general, the number of positive integer solutions to $\sum_{i=1}^{k} x_i = n$ is $\binom{n-1}{k-1}$.

Example: How many ways can I distribute 12 pieces of candy to 3 of my friends, such that they all get at least one piece?

This is the same problem as finding the number of positive integer solutions to $x_1 + x_2 + x_3 = 12$ which from the previous slide we have as

$$\binom{12-1}{3-1} = \underbrace{\binom{11}{2}}_{\binom{n-1}{k-1}} = 55$$

from last slide.

Another way to think of this constrained problem: Since all of my friends will get at least one piece, I can give them each one piece to begin with, and then figure out the number of ways to distribute the remaining $12 - 3 = \underline{9}$ pieces, which is $\binom{9+2}{2}$.

stars = 9

bars = 2

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\binom{n+1}{2} = \frac{(n+1)!}{2!\,(n-1)!} = \frac{(n+1)\,n\,\cancel{(n-1)!}}{2\,\cancel{(n-1)!}}$$

Pascal's Triangle

Hockey-stick

$\binom{r}{1}$

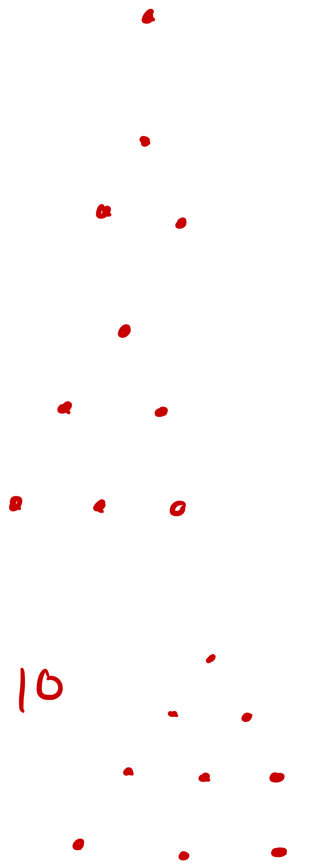
$\binom{r}{2}$

1 : 1

2 : 3

3 : 6

4 : 10



$\binom{3}{2}$

$\binom{4}{2}$

$\binom{5}{2}$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

$\binom{r}{2}$

- natural #s
- symmetric

triangular #s

sum of $1+2+\dots+n = \binom{n+1}{2}$

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$