Lecture 18: Pascal's Triangle and Combinatorial Proofs

Introduction to Mathematical Thinking

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Announcements

- Today:
    1. Finishing up Pascal's Triangle and Combinatorial Proofs
    2. Walking through problems from Homework 6
- Quiz 4 on Tuesday, 3:40-4:10
    - Covers everything since Quiz 3, up to and including today
- Grades in the course
    - After Quiz 4, I will tally up everyone's total number of points in the course and determine the number of points required to pass

Pascal's Triangle

1 $\to 2^0$

1 1 $\to 2^1$

1 2 1 $\to 2^2$

1 3 3 1 $\to 2^3$

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

$$
\begin{array}{c}
\binom{0}{0} \quad \text{row } 0 \\
\binom{1}{0} \quad \binom{1}{1} \\
\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2} \\
\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3} \\
\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} \\
\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}
\end{array}
$$

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

Defining property (Pascal's identity):

$$\binom{n}{k} + \binom{n \cancel{+1}}{k+1} = \binom{n+1}{k+1}$$

Some properties:

- $n+1$ properties in row n
- Rows are symmetric
- "Hockey-Stick"
- Third diagonal: triangular numbers
- Sum of nth row: 2^n

Combinatorial Proofs

ex

$S = \{a, b, c\}$

$3: \{a, b, c\}$ $\binom{3}{3}$

$2: \{a, b\}, \{a, c\}, \{b, c\}: \binom{3}{2}$

$0: \varnothing$ $\binom{3}{0}$

$1: \{a\}, \{b\}, \{c\}$ $\binom{3}{1}$

Instead of proving a statement algebraically, we can prove statements *combinatorially*, by showing that both sides of the equals sign count the same quantity.

For example: Let's give a combinatorial proof of the fact that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

i.e. that the sum of the nth row of Pascal's Triangle is 2^n.

$|S| = n$

- LHS: Count the number of subsets of a set of size n. We can either choose 0 elements, or 1 element, or 2 elements, ... , or n elements. We can choose k elements in $\binom{n}{k}$ ways, therefore we have $\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$.
- RHS: For each of the n elements, we have two choices - it is either included in our subset, or not. This yields 2^n total options.

$\underset{s_1}{\underline{2}}\ \underset{s_2}{\underline{2}} \cdots \underline{2} \cdots \underline{2}\ \underset{s_n}{\underline{2}} = 2^n$

Thus, $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n-1} + \binom{n}{n} = 2^n$, and the sum of the nth row of Pascal's Triangle is 2^n.

Example: Prove $\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + mn$.

Suppose we have m Warriors fans and n Lakers fans.

LHS: Number of ways to select 2 basketball fans from the set of $m+n$ is $\binom{m+n}{2}$.

RHS: To select 2 basketball fans from our set of $m+n$, we either take
- 2 Warriors fans, $\binom{m}{2}$ or ✓
- 2 Lakers fans, $\binom{n}{2}$ or ✓
- 1 Warriors fan and 1 Lakers fan, $\binom{m}{1}\binom{n}{1} = mn$

Both the LHS and RHS count the same quantity, therefore we must have
$\binom{m+n}{2} = \binom{m}{2} + \binom{n}{2} + mn$.

Pascal's Identity

Formalization of the fact that the sum of two adjacent numbers in the triangle is the number directly below them.

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Can you think of a combinatorial proof for this?

Algebraic Proof

$$\binom{n}{k} + \binom{n}{k+1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!}$$

$$= \frac{n!(k+1)}{(k+1)!(n-k)!} + \frac{n!(n-k)}{(k+1)!(n-k)!}$$

$$= \frac{n!(k+1+n-k)}{(k+1)!(n+k)!} = \frac{(n+1)!}{(k+1)!(n+k)!}$$

$$= \binom{n+1}{k+1}$$

$p_1, p_2, \dots, p_n, p_{n+1}$

Combinatorial Proof

1) p_1 included
n remaining $\binom{n}{k}$
k select

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

RHS: Number of ways to choose $k+1$ people from a group of $n+1$.

2) p_1 not included
n remaining $\binom{n}{k+1}$
$k+1$ select

LHS: Suppose we want to choose $k+1$ people from a group of $n+1$. Suppose the people are numbered $p_1, p_2, \dots p_{n+1}$. Consider the very first person: either we include them in our subset or do not include them.

- If we include them, there are n people remaining and we need to choose k of them: $\binom{n}{k}$
- If we do not include them, there are n people remaining and we need to choose $k+1$ of them: $\binom{n}{k+1}$

Thus, the total number of ways to choose $k+1$ people from a group of $n+1$ is $\binom{n}{k} + \binom{n}{k+1}$. We've already shown this quantity is $\binom{n+1}{k+1}$, though, so these expressions both must be the same!

Example: Give a combinatorial proof of Vandermonde's Identity, that is:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \left(\binom{m}{k} \binom{n}{r-k} \right)$$

Hint: Think of Warriors/ Lakers example

$r \leq \min(m, n)$

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

LHS: Number of ways to choose r basketball fans from m Warriors fans and n Lakers fans

RHS: Suppose we want to choose r basketball fans from m Warriors fans and n Lakers fans. If we choose k Warriors fans, we need to choose $r-k$ Lakers fans. The total number of fans we choose must always be r, and this value of k can be anything from 0 to r.

We could choose 0 Warriors fans and r Lakers fans, or 1 and $r-1$, or 2 and $r-2$, ..., or $r-1$ and 1, or r and 0, giving us $\underline{\binom{m}{0}\binom{n}{r}} + \underline{\binom{m}{1}\binom{n}{r-1}} + \ldots + \underline{\binom{m}{r}\binom{n}{0}} = \sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}$, as required.

Corollary: $\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2$

$m = n = r$

$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2$