Announcements

- Homework 7 will come out later this week, and be due **this Sunday**
    - It will be very short (~3 problems), since you will have less than a week to work on it
    - Mainly on concepts from today's lecture, which we will cover again on Thursday
- Will start consolidating grades
- Pass threshold will likely change

Now:

- Make sure to read the Binomial Theorem note on the website.

Binomial Theorem

Binomial: A polynomial with two terms, joined by addition.

$$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd$$

When multiplying two binomials, the result is every combination of one term in the first binomial multiplied by one term in the second binomial.

$$(x + y)^2 = (x + y)(x + y) = xx + xy + yx + yy = x^2 + 2xy + y^2$$

1 2 1

$(x+y)^3$

$\binom{3}{0}x^3$
$\binom{3}{1}x^2y$
$\binom{3}{2}xy^2$
$\binom{3}{3}y^3$

$(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$

$$(x+y)^2 = (x+y)(x+y) = xx + xy + yx + yy = x^2 + 2xy + y^2$$

$\binom{2}{0}x^2$
$\binom{2}{1}xy$
$\binom{2}{2}y^2$

Either we choose...

- 2 xs and 0 ys: $\binom{2}{0}$
- 1 x and 1 y: $\binom{2}{1}$
- 0 xs and 2 ys: $\binom{2}{2}$

Now, we have $\binom{2}{0}$ terms of the form x^2, $\binom{2}{1}$ terms of the form xy and $\binom{2}{2}$ terms of the form y^2:

$$\underline{(x+y)^2} = \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2$$

To generalize: Each term in the expansion of $(x+y)^n$ has k xs and $n-k$ ys, for $k = 0, 1, \ldots n$.

Formalization of the Binomial Theorem

The binomial theorem states

choosing k ys,
n-k x s

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$= \binom{n}{0} x^k + \binom{n}{1} x^{k-1} y + \binom{n}{2} x^{k-2} y^2 + \ldots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

For example, let's expand $(2a^2 - b)^5$.

$x = 2a^2$

$y = -b$

$$(2a^2 - b)^5 = \binom{5}{0} x^5 + \binom{5}{1} x^4 y + \binom{5}{2} x^3 y^2 + \cdots + \binom{5}{5} y^5$$

$$= \binom{5}{0}(2a^2)^5 + \binom{5}{1}(2a^2)^4(-b) + \binom{5}{2}(2a^2)^3(-b)^2 + \cdots + \binom{5}{5}(-b)^5$$

$$= 1 \cdot 32a^{10} - 80 \cdot a^8 b + \cdots + (-b^5)$$

General Term

$$(x+y)^n = \sum_{k=0}^{n} t_k$$

We define the k-th term in the expansion of a binomial as

$$t_k = \binom{n}{k} x^{n-k} y^k$$

with $k \in \{0, 1, 2, ..., n\}$.

$3^{5-k} \quad a^{10-2k}$

General term of $(3a^2 - 2b)^5$:

$x = 3a^2$

$y = -2b$

$$\begin{aligned} t_k &= \binom{5}{k} (3a^2)^{5-k} (-2b)^k \\ &= \binom{5}{k} 3^{5-k} a^{10-2k} (-1)^k 2^k b^k \\ &= (-1)^k \binom{5}{k} 2^k 3^{5-k} \underline{a^{10-2k}}\, \underline{b^k} \end{aligned}$$

Example: What is the general term of $(x^5 - \frac{1}{x^2})^7$?

$$= (x^5 - x^{-2})^7$$

$$(-x^{-2})^k = (-1)^k x^{-2k}$$

$$t_k = \binom{7}{k} (x^5)^{7-k} (-x^{-2})^k$$

$$= (-1)^k \binom{7}{k} x^{35-5k} x^{-2k}$$

$$= \boxed{(-1)^k \binom{7}{k} x^{35-7k}}$$

Now: Applications and Properties of the Binomial Theorem

Example: Sum of Coefficients

What is the sum of the coefficients of $(3x^2 - 4x)^{12}$?

$$(3x^2-4x)^{12} = \binom{12}{0}(3x^2)^{12} + \binom{12}{1}(3x^2)^{11}(-4x) + \cdots +$$

$\Downarrow$ $\underline{3^{12}}$

$\Downarrow$ $\underline{12 \cdot 3^{11} \cdot (-4)}$

$\Downarrow$

$x=1$

$$(3-4)^{12} = (-1)^{12}$$

$$= \boxed{1}$$

$\Rightarrow$ let all variables be $\underline{1}$.

$$f(x,y,z) = 3xy^{15} - 17x^2yz$$

$$f(1,1,1) = 3-17 = -14$$

Example: Sum of the nth row of Pascal's Triangle

Previously, we proved that the sum of the nth row of Pascal's Triangle is 2^n using a combinatorial argument. How can we do this using the Binomial Theorem?

$$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Let $x = y = 1$

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$$

$$2^n = \dots$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Example: Approximations

We know that $\binom{n}{k}$ is only defined for whole numbers n, k, such that $n \geq k$. This is because $n!$ is only defined for whole n.

However, we can rewrite $\binom{n}{k}$ to not use any factorials.

$$\binom{n}{0} = 1$$

$$\binom{n}{1} = n$$

$$\binom{n}{2} = \frac{n\cdot(n-1)\cdot\cancel{(n-2)!}}{2!\,\cancel{(n-2)!}} = \frac{n(n-1)}{2!}$$

$$\binom{n}{3} = \dots = \frac{n(n-1)(n-2)}{6 = 3!}$$

$$\sqrt{9.02} = (\underset{x}{9} + \underset{y}{0.02})^{\frac{1}{2}\; n}$$

$$= \binom{n}{0} 9^n + \binom{n}{1} 9^{n-1}\, 0.02^1 + \binom{n}{2} 9^{n-2}\, 0.02^2 + \dots$$

$$= 1\cdot 9^{1/2} + \binom{1/2}{1} 9^{-1/2}\, 0.02 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2} 9^{-\frac{3}{2}}\, 0.02^2$$

$$= 3 + \cancel{\frac{1}{2}\cdot\frac{1}{3}\cdot 0.02}_{0.03} - \frac{1}{8}\cdot\frac{1}{27}\cdot 0.02^2$$

$$(x+y)^p = 1x^p + \underbrace{\binom{p}{1}x^{p-1}y + \binom{p}{2}x^{p-2}y^2 + \cdots + \binom{p}{p-1}xy^{p-1}} + y^p$$

Show $\binom{p}{i} \equiv 0 \mod p$ when $i = 1, 2, \ldots p-1$

Example: Proof of Freshman's Dream

The freshman's dream identity states

$$(x+y)^p \equiv x^p + y^p \pmod p$$

for a prime p. How can we use the Binomial Theorem to help us prove this?

$$\binom{p}{i} = \frac{p!}{i!(p-i)!}$$

$p!$ → num is multiple of p

$i!(p-i)!$ → since p is prime, denominator contains no factor/multiple of p

$$\Rightarrow \binom{p}{i} = cp, \quad c \in \mathbb{Z}^+$$