Lecture 20: Binomial Theorem, Vieta's Formulas

Introduction to Mathematical Thinking

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Announcements

- Homework 7 is out, due Sunday 11:59pm
    - Only 3 problems long, you can even think of it as a "vitamin"
    - More involved homework on these topics will be due following week

Today: Will finish talking about the Binomial Theorem, and start talking about Vieta's formulas. There is also a note on the website for Vieta's; would again highly recommend a read.

$(x+y)^2 = 1x^2 + 2xy + 1y^2$

Formalization of the Binomial Theorem

The binomial theorem states

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$= \binom{n}{0} x^k + \binom{n}{1} x^{k-1} y + \binom{n}{2} x^{k-2} y^2 + \ldots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

We define the k-th term, i.e. the **general term**, in the expansion of a binomial as

$$t_k = \binom{n}{k} x^{n-k} y^k$$

with $k \in \{0, 1, 2, \ldots, n\}$.

Example: Sum of Coefficients

What is the sum of the coefficients of $(3x^2 - 4x)^{12}$?

$x = 1$

$(3 \cdot 1^2 - 4 \cdot 1)^{12} = (-1)^{12} = 1$

Example: Sum of the nth row of Pascal's Triangle

Previously, we proved that the sum of the nth row of Pascal's Triangle is 2^n using a combinatorial argument. How can we do this using the Binomial Theorem?

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^{k}$$

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Example: Approximations

We know that $\binom{n}{k}$ is only defined for whole numbers n, k, such that $n \geq k$. This is because $n!$ is only defined for whole n.

However, we can rewrite $\binom{n}{k}$ to not use any factorials.

$$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$$

$$\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{n(n-1)(n-2)}{6}$$

can pass in any n, not just wholes

Now, for example, to compute $\sqrt[3]{8.03}$:

$$\begin{aligned} 8.03^n &= (8 + 0.03)^n \\ &= 8^n + n \cdot 8^{n-1} \cdot 0.03 + \frac{n(n-1)}{2} 8^{n-2} \cdot 0.03^2 \\ &= 8^{\frac{1}{3}} + \frac{1}{3} \cdot 8^{-\frac{2}{3}} \cdot 0.03 + \frac{\frac{1}{3} \cdot (-\frac{2}{3})}{2} 8^{-\frac{5}{3}} \cdot 0.03^2 \\ &= 2.002496875 \end{aligned}$$

"$\binom{n}{2}$"

Calculator yields 2.00249688.

Example: Proof of Freshman's Dream

The freshman's dream identity states

$$(x+y)^p \equiv x^p + y^p \pmod p$$

for a prime p. How can we use the Binomial Theorem to help us prove this?

$$(x+y)^p = x^p + \underbrace{\binom{p}{1} x^{p-1} y + \cdots + \binom{p}{p-1} x y^{p-1}}_{\text{each term contains a multiple of } p} + y^p$$

$$\binom{p}{i} \equiv 0 \mod p$$

when $i \neq 0, p$

p prime

Example: Determining Coefficients without Expansion

Suppose we want to determine the coefficient of x^{20} in $(x^5 - 5)^7$. How can we use the general term to help us?

$$t_k = \binom{7}{k}(x^5)^{7-k}(-5)^k$$

$$= (-1)^k \binom{7}{k} 5^k x^{35-5k}$$

Take exp on x, and set it to 20

$35 - 5k = 20$

$\Rightarrow k = 3$

$$t_3 = (-1)^3 \binom{7}{3} 5^3 x^{20}$$

$$t_k = (-1)^k \binom{7}{k} 5^k x^{35-5k}$$

Coefficient on x^{21} ?

$35 - 5k = 21$

$5k = 14$

$k = \frac{14}{5}$

k must be $\in \mathbb{N}_0$

$\therefore$ no term with x^{21}

$\therefore$ coefficient $(x^{21}) = 0$

Example: Re-writing Sums

Suppose we have

$$f(x) = \sum_{k=0}^{7} (-1)^k (7-k) \binom{7}{k} x^{6-k}$$

Determine $f(3)$.

$$\frac{d}{dx} c x^{7-k} = c(7-k) x^{6-k}$$

Recall:

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$= \sum_{k=0}^{7} \binom{7}{k} (7-k) x^{6-k} (-1)^k$$

$$= \frac{d}{dx} \sum_{k=0}^{7} \binom{7}{k} x^{7-k} (-1)^k$$

$$= \frac{d}{dx} (x-1)^7 = 7(x-1)^6$$

$$f(3) = 7 \cdot (3-1)^6$$

$$= 64 \cdot 7$$

$$= 448$$

Trinomial Theorem?

Suppose we want to expand $(x+y+z)^n$. We *could* treat $x+y$ as a single term and use the binomial expansion...

$$(x+y+z)^n = ((x+y)+z)^n$$

$$= \binom{n}{0}(x+y)^n + \binom{n}{1}(x+y)^{n-1}z + \ldots + \binom{n}{n-1}(x+y)z^{n-1} + \binom{n}{n}z^n$$

However, we would then need to expand each term $(x+y)^i$ again with the binomial theorem... that's messy.

$$(x+y+z)(x+y+z)(x+y+z)$$

Suppose a general term in the expansion of $(x+y+z)^n$ contains a xs, b ys and c zs. Then, we must have that $a+b+c=n$, since the total number of parentheses we choose from in the expansion must be exactly n. Then:

$$t_{a,b,c} = \frac{n!}{a!b!c!}x^a y^b z^c$$

The coefficient $\frac{n!}{a!b!c!}$ comes from the number of ways to arrange a xs, b ys and c zs (think back to counting the number of permutations of MISSISSIPPI).

$$\binom{N}{a}\binom{N-a}{b}\binom{N-a-b}{c} = \frac{N!}{a!(N-a)!}\cdot\frac{(N-a)!}{b!(N-a-b)!}\cdot\frac{(N-a-b)!}{c!(N-a-b-c)!}$$

$$= \frac{N!}{a!b!c!}$$

Example: Calculate the coefficient of x^4 in the expansion of $(x - 3x^{-2} + 4)^8$.

$$t_{a,b,c} = \frac{8!}{a!b!c!} x^a (-3x^{-2})^b (4)^c$$
$$= (-1)^b \frac{8!}{a!b!c!} 3^b 4^c x^{a-2b}$$

We need $a - 2b = 4$, with the constraints $0 \leq a, b, c \leq 8$ and $a + b + c = 8$. With some trial and error, we can identify the only two solutions, $(4, 0, 4)$ and $(6, 1, 1)$.

Then:

$$t_{4,0,4} = (-1)^0 \frac{8!}{4!0!4!} 3^0 4^4 x^4 = 17920x^4$$
$$t_{6,1,1} = (-1)^1 \frac{8!}{6!1!1!} 3^1 4^1 x^4 = -336x^4$$

Thus, the coefficient on x^4 is $17920 - 336 = 17584$.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^{k}$$

Generalization of the "Trinomial Theorem"

$$(x+y+z)^n = \sum_{a,b,c:a+b+c=n} \frac{n!}{a!b!c!} x^a y^b z^c$$

This is similar to the way we can represent the binomial theorem:

$$(x+y)^n = \sum_{a,b:a+b=n} \frac{n!}{a!b!} x^a y^b$$

a	b
n	0
n − 1	1
n − 2	2
⋮	⋮
0	n

However, this expression of the "trinomial" theorem is less meaningful, as there's no easy way to express this sum any simpler.

Multinomial Theorem

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$$\binom{11}{1,4,4,2} = \frac{11!}{1!4!4!2!}$$

We can further define the "multinomial" coefficient:

$$\binom{n}{k_1, k_2, ...k_m} = \frac{n!}{k_1!k_2!...k_m!}$$

Under the assumption $k_1 + k_2 + ... + k_m = n$, this term is the number of ways to select one subset of size k_1, one subset of size k_2, ... and one subset of size k_m from a group of n items.

$$\binom{n}{k_1} \cdot \binom{n-k_1}{k_2} \cdot \binom{n-k_1-k_2}{k_3} \cdot ... \cdot \binom{n-k_1-k_2-...-k_{m-1}}{k_m}$$

$$= \frac{n!}{k_1!k_2!k_3! \cdot ... \cdot k_m!}$$

$$\binom{n}{k_1, k_2} = \frac{n!}{k_1!k_2!} = \frac{n!}{k_1!(n-k_1)!} = \binom{n}{k_1}$$

For example, $\binom{11}{1,4,4,2}$ is the number of permutations of MISSISSIPPI (we choose 1 character to be an M, 4 to be an I, 4 to be an S and 2 to be a P).

$= \sum \binom{n}{k_1, k_2, \ldots k_m} x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}$

Then:

m^n

$$(x_1 + x_2 + \ldots + x_m)^n = \sum_{k_1+k_2+\ldots+k_m=n} \binom{n}{k_1, k_2, \ldots k_m} \prod_{i=1}^{m} x_i^{k_i}$$

This last expansion is that of the "multinomial" theorem!

Question: What is the sum of all multinomial coefficients of m terms? (Hint: With $m = 2$, what is this quantity?)

with a fixed n

$\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

$$a_i \in \mathbb{R}$$

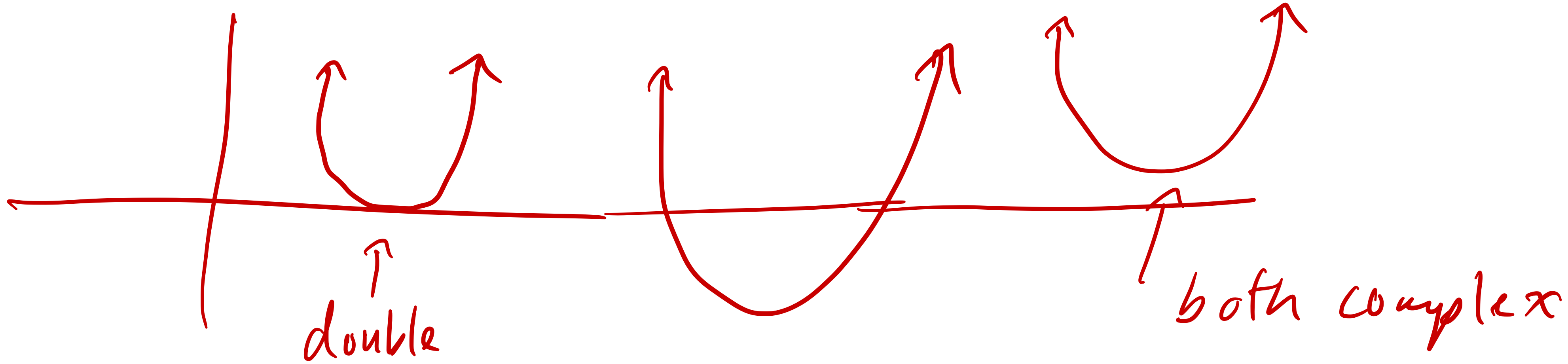
Vieta's Formulas

(Recall: A polynomial of degree n has exactly n roots, some of which may be the same, and some of which may be complex. The notes talk more about this.)

Vieta's formulas give us a way to interpret a polynomial in standard form, e.g. $p(x) = ax^2 + bx + c$, in terms of its roots, without having to find the roots specifically.

In the above $p(x)$: what is the sum of the roots? The product?

One way to determine: Use the quadratic formula to solve for both roots, and simplify.



$p(x) = ax^2 + bx + c$

2 roots, r_1, r_2

Using the quadratic formula:

$$r_1, r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Then:

$$r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a} = -\frac{2b}{2a} = -\frac{b}{a}$$

$$r_1 r_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

It works! How would we extend this to cubic polynomials, though?

There's a simpler way to look at this.

$p(x) = 2x^2 - 7x + 15$

sum : $\frac{7}{2}$

prod : $\frac{15}{2}$

ax^2+bx+c

Suppose $p(x)$ has 2 roots, r_1, r_2.

$$p(x) = a(x-r_1)(x-r_2)$$

$$= a\left[x^2 + x(-r_2) + x(-r_1) + (-r_1)(-r_2)\right]$$

2 xs, 0 rs

1 xs 1 rs

0 xs, 2 rs

$$= a\left[x^2 - (r_1+r_2)x + r_1 r_2\right]$$

$$= ax^2 - a(r_1+r_2)x + a r_1 r_2$$

$$ax^2 \quad + \quad b \quad x \quad + \quad c$$

$$b = -a(r_1+r_2)$$

$$\Rightarrow \boxed{r_1 + r_2 = -\frac{b}{a}}$$

$$c = a r_1 r_2$$

$$\Rightarrow \boxed{r_1 r_2 = \frac{c}{a}}$$

Suppose $p(x) = ax^2 + bx + c$ has two roots, r_1 and r_2. Then:

$$p(x) = a(x - r_1)(x - r_2) = ax^2 - a(r_1 + r_2)x + ar_1r_2$$

By comparison, we can see $b = -a(r_1 + r_2)$ and $c = ar_1r_2$, i.e.

$$r_1 + r_2 = -\frac{b}{a}$$

$$r_1r_2 = \frac{c}{a}$$

This is the same result we found before, using the quadratic formula.

These are **Vieta's formulas for degree-2 polynomials**.

In $p(x) = 4x^2 + 3x + 3$, we see that the sum of the roots is $-\frac{3}{4}$ and product is $\frac{3}{4}$.

Cubic ($d=3$)

$$p(x) = a(x-r_1)(x-r_2)(x-r_3)$$

$$p(x) = ax^3 + bx^2 + cx + d$$

$$p(x) = a\left[x^3 + x^2(-r_1 - r_2 - r_3) + x(r_1r_2 + r_1r_3 + r_2r_3) + (-r_1)(-r_2)(-r_3)\right]$$

3 xs, 0 roots

2 xs, 1 roots $\qquad \binom{3}{2} = 3$

1 x, 2 roots

0 xs, 3 roots

$$= a\left(x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3\right)$$

symmetric sum

$p(x) = 3x^3 - 14x^2 + 7x - 12$

sum: $\frac{14}{3}$ prod: $-\frac{-12}{3} = 4$

What about for cubic polynomials, of the form $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$? Let's assume $p(x)$ has roots r_1, r_2, r_3. Then:

$$a_3x^3 + a_2x^2 + a_1x + a_0 = a_3(x - r_1)(x - r_2)(x - r_3)$$

We have the following choices in this expansion:

- We can choose 3 xs and no roots, yielding x^3
- We can choose 2 xs and one root, yielding $(-r_1 - r_2 - r_3)x^2 = -(r_1 + r_2 + r_3)x^2$
- We can choose 1 x and two roots, yielding $((-r_1) \cdot (-r_2) + (-r_1) \cdot (-r_3) + (-r_2) \cdot (-r_3))x = (r_1r_2 + r_1r_3 + r_2r_3)x$
- We can choose no xs and three roots, yielding $((-r_1) \cdot (-r_2) \cdot (-r_3)) = -r_1r_2r_3$

This gives $r_1 + r_2 + r_3 = -\frac{a_2}{a_3}$, $r_1r_2 + r_1r_3 + r_2r_3 = \frac{a_1}{a_3}$, and $r_1r_2r_3 = -\frac{a_0}{a_3}$. Note the alternating signs.

Each successive term is a **sum of products of roots, taken in different quantities at a time**.

- $-\frac{a_2}{a_3} = r_1 + r_2 + r_3$ is the sum of the products of the roots, taken one at a time, since multiplying a constant by nothing is the constant itself.
  - There are $\binom{3}{1} = 3$ terms in this sum
- $\frac{a_1}{a_3} = r_1r_2 + r_1r_3 + r_2r_3$ is the sum of the product of the roots, taken two at a time â€" it features all 3 possible combinations of two different roots multiplied together.
  - There are $\binom{3}{2} = 3$ terms in this sum
- $-\frac{a_0}{a_3} = r_1r_2r_3$ is the sum of the product of the roots, taken three at a time â€" there is only one way to take three items at once, and this is that one way.
  - There are $\binom{3}{3} = 1$ terms in this sum

Exercise: Without manual expansion, determine Vieta's formulas for polynomials of degree 4.

$$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = a_4(x - r_1)(x - r_2)(x - r_3)(x - r_4)$$

$$-\frac{a_3}{a_4} = r_1 + r_2 + r_3 + r_4 \qquad \binom{4}{1}$$

$$\frac{a_2}{a_4} = r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 \qquad \binom{4}{2}$$

$$-\frac{a_1}{a_4} = r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 \qquad \binom{4}{3}$$

$$\frac{a_0}{a_4} = r_1r_2r_3r_4$$

We can generalize this to n-degree polynomials!

Generalized Vieta's Formulas

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

$$= a_n \sum_{k=0}^{n} (-1)^k \left(\text{sum of the products of the roots of } p(x)\text{, taken } k \text{ at a time}\right) x^{n-k}$$

The algebraic definition isn't as important. What's more important is identifying this pattern.

$$p(x) = 540072^{97} x^{100} + 0x^{99} + x^{98} - 17x^{92} + \ldots + x^{15} - 3$$

The binomial theorem is actually just a special case of Vieta's formulas, when all roots are the same! For example, suppose $n = 4$, $r_i = c$ for all i and the leading coefficient is 1. Then:

$$p(x) = (x - c)^4 = \binom{4}{0}x^4 - \binom{4}{1}x^3c + \binom{4}{2}x^2c^2 - \binom{4}{3}xc^3 + \binom{4}{4}c^4$$

Using Vieta's formulas for $n = 4$:

$$a_3 = -(r_1 + r_2 + r_3 + r_4) = -4c$$

$$a_2 = r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 = 6c^2$$

$$a_1 = -(r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4) = -4c^3$$

$$a_0 = r_1r_2r_3r_4 = c^4$$

Example: Suppose a, b satisfy $x^2 - 18x + 18 = 0$. Determine $a^2 + b^2$.

$$a + b = 18$$

$$a \cdot b = 18$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^2 - 2ab = a^2 + b^2$$

$$18^2 - 2 \cdot 18 = a^2 + b^2$$

$$\boxed{288 = a^2 + b^2}$$

$+0x^2$

Example: $p(x) = x^3 - Ax + 15$ has three real roots, two of which sum to 5. What is $|A|$?

$r_1 + r_2 = 5$

$r_3 = -5$

Example: $p(x) = x^3 - Ax + 15$ has three real roots, two of which sum to 5. What is $|A|$?

Solution: Let r_1, r_2 be the roots that sum to 5. This must mean $r_3 = -5$, since $r_1 + r_2 + r_3 = 5 - 5 = 0$ (there is no x^2 term).

$r_1 r_2 = \frac{r_1 r_2 r_3}{r_3}$

We also know $r_1 r_2 r_3 = -15$. Then,

$$-A = r_1 r_2 + r_1 r_3 + r_2 r_3 = r_3(r_1 + r_2) + r_1 r_2$$

$$= r_3(r_1 + r_2) + \frac{r_1 r_2 r_3}{r_3}$$

$$= -5(5) + \frac{-15}{-5} = -25 + 3 = -22$$

Thus, $|A| = 22$.