Lecture 21: Vieta's Formulas

Introduction to Mathematical Thinking

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Announcements

- You should have received a grading breakdown yesterday via email.
  - New pass threshold is 65
  - **Attendance:** There are 15 possible check-ins at this point (this doesn't include the very first lecture, or most of the lectures I was not here for)
    - If your check-in count is 11 or fewer, you will have to do an extra-credit assignment to make up for it
    - If your count is 12 or more, and you don't have any more unexcused absenses in the next two weeks, you're fine
    - Current counts don't include excused absences or waitlist acceptances (who will have missed the first meeting), so if your count is 11 and you had an excused absence, please reach out
- Assignments left: Quiz 5 (2 weeks from today), Homework 8 (will be due right before Quiz 5)
- Today: Mostly problems. Other fun stuff with polynomials in the next class, then review next week.

ax^2+bx+c

$r_1+r_2=-\frac{b}{a}$ $r_1\cdot r_2=\frac{c}{a}$

Vieta's Formulas

Degree-2 case: $p(x) = a_2x^2 + a_1x + a_0$

$$r_1 + r_2 = -\frac{a_1}{a_2} \qquad r_1r_2 = \frac{a_0}{a_2}$$

Degree-3 case: $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

$$\underbrace{r_1 + r_2 + r_3}_{\binom{3}{1}=3 \text{ terms in sum}} = -\frac{a_2}{a_3} \qquad \underbrace{r_1r_2 + r_1r_3 + r_2r_3}_{\binom{3}{2}=3 \text{ terms in sum}} = \frac{a_1}{a_3} \qquad \underbrace{r_1r_2r_3}_{\binom{3}{3}=1 \text{ term in sum}} = -\frac{a_0}{a_3}$$

Generalize?

Generalized Vieta's Formulas

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

$$= a_n \sum_{k=0}^{n} (-1)^k \big(\text{sum of the products of the roots of } p(x) \text{, taken } k \text{ at a time}\big) x^{n-k}$$

$(x-r_1)(x-r_2)(x-r_3)(x-r_4) = x^4 - (r_1+r_2+r_3+r_4)x^3$ $\binom{4}{1}$ terms $\binom{4}{2}$ terms

$+ (r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_1r_4)x^2$ $\binom{4}{3}$ terms

$- (r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4)x$

$+ r_1r_2r_3r_4$

$r_1r_2r_3 = \dfrac{r_1r_2r_3r_4}{r_4}$

$r_1r_2r_4 = \dfrac{r_1r_2r_3r_4}{r_3}$

$r_1r_3r_4 = \dfrac{r_1r_2r_3r_4}{r_2}$

Sum of roots:

$$r_1 + r_2 + \ldots + r_n = -\frac{a_{n-1}}{a_n}$$

Product of roots:

$$r_1 \cdot r_2 \cdot \ldots \cdot r_n = (-1)^n \frac{a_0}{a_n}$$

Example: Let $f(x) = (x^2 + 6x + 9)^{50} - 4x + 3$, and suppose $f(x)$ has 100 roots, $r_1, r_2, r_3, \ldots, r_{100}$. Determine $(r_1 + 3)^{100} + (r_2 + 3)^{100} + \ldots + (r_{100} + 3)^{100}$.

$$f(x) = (x^2+6x+9)^{50} - 4x + 3 = 0$$

$$(x+3)^{100} = 4x - 3$$

for any root r_i, $(r_i+3)^{100} = 4r_i - 3$

sum both sides $\sum_{i=1}^{100} (r_i+3)^{100} = \sum_{i=1}^{100} (4r_i - 3)$

$$= \sum_{i=1}^{100} 4r_i - 300$$

$$= 4\sum_{i=1}^{100} r_i - 300$$

$$= 4(-300) - 300$$

$$= \boxed{-1500}$$

sum of roots of $f(x)$ =

sum of roots of $(x+3)^{100}$

$= -(\text{coeff on } x^{99})$

$= -\left(\binom{100}{1} \cdot 3\right) = -300$

$x^3 + 7x^2 - 15x$

$x^3 + 7x^2 + 200^{200}x - 155$

$x^3 + 7x^2 - 13x + 4$

$x^k + a_{k-1}x^{k-1} + \dots + a_2x^2 + a_1x + a_0$

Example: Suppose we have, for some odd integer k,

$$f_k(x) = \prod_{i=1}^{k}(x-i) = (x-1)(x-2)(x-3)\cdots(x-k)$$

a) Determine the coefficient on x^{k-1}.

b) Determine the coefficient on x.

Recall

$$1+2+\dots+n = \frac{n(n+1)}{2}$$

a) roots are $1, 2, \dots k$

$\sum \text{roots} = -a_{k-1}$

$a_{k-1} = -\sum \text{roots}$

$= -\sum_{i=1}^{k} i$

$= -\frac{k(k+1)}{2}$

b) next slide

b)

$$f_k(x) = (x-1)(x-2)\cdots(x-(k-1))(x-k)$$

coefficient on x

$$\hookrightarrow \left(r_1 r_2 r_3 \cdots r_{k-1} + r_1 r_2 \cdots r_{k-2} r_k + r_1 r_2 \cdots r_{k-3} r_{k-1} r_k + \cdots + r_1 r_3 \cdots r_k + r_2 r_3 \cdots r_k\right)$$

$$= r_1 r_2 r_3 \cdots r_k \left(\frac{1}{r_k} + \frac{1}{r_{k-1}} + \cdots + \frac{1}{r_2} + \frac{1}{r_1}\right)$$

$$\Rightarrow \ = k!\left(\frac{1}{k} + \frac{1}{k-1} + \cdots + \frac{1}{2} + 1\right)$$

$$= \boxed{(k!) \sum_{i=1}^{k} \frac{1}{i}}$$

$\Downarrow$ $\boxed{\beta^6 - \alpha\beta^4 + \beta = 0}$

Example: Suppose the polynomial $x^3 - \alpha x^2 + \beta$ has three roots, one of which is equal to β^2.

a) What is the sum of all possible values of β? 0

b) What is the product of all non-zero possible values of β? β

(Hint: Is using Vieta's formulas the easiest way to do this?)

Vieta's

Let $r_3 = \beta^2$.

$r_1 + r_2 + \beta^2 = \alpha$

$r_1 r_2 + r_1 r_3 + r_2 r_3 = 0$

$r_1 r_2 r_3 = -\beta$

$r_1 r_2 + r_3(r_1 + r_2) = 0$

$= \frac{r_1 r_2 r_3}{r_3} + r_3(r_1 + r_2) = 0$

$r_1 + r_2 = \alpha - \beta^2$ ✓

$r_1 r_2 + r_3(r_1 + r_2) = 0$

$\frac{r_1 r_2 r_3}{r_3} + r_3(r_1 + r_2) = 0$

$\frac{-\beta}{\beta^2} + \beta^2(\alpha - \beta^2) = 0$

$-\beta + \beta^4(\alpha - \beta^2) = 0$

$-\beta + \alpha\beta^4 - \beta^6 = 0$

$\Rightarrow \boxed{\beta^6 - \alpha\beta^4 + \beta = 0}$

$\therefore$ sum of all possible $\beta = 0$

Assume $a_i \in \mathbb{R}$

Example: Consider $p(x) = a_2x^2 + a_1x + a_0$ with roots r_1, r_2. Prove, using Vieta's formulas, that if r_1 is complex, then r_2 is the complex conjugate of r_1.

$z = a + bi,\quad a, b \in \mathbb{R}$

$\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\}$

If $z = a + bi,$

$\bar{z} = a - bi$

Assume $r_1 = a + bi$

$r_2 = c + di,$

show $a = c,$

$b = -d$

$\Rightarrow$ if $r_1 = a + bi,$ then

$r_2 = a - bi$

$r_1 + r_2 = -\dfrac{a_1}{a_2} \in \mathbb{R}$

$a+bi+c+di$

$= (a+c) + (b+d)i = -\dfrac{a_1}{a_2}$

$\Rightarrow b+d=0$

$\Rightarrow \boxed{b = -d}$

$r_1 \cdot r_2 = \dfrac{a_0}{a_2} \in \mathbb{R}$

$(a+bi)(c+di)$

$= ac + adi + bci + bdi^2$

$= ac - bd + (ad + bc)i \in \mathbb{R}$

$ad + bc = 0$

$a(-b) + bc = 0$

$(a-c)b = 0$

$\Rightarrow \boxed{a = c}$

Reciprocal Polynomials

r is a root of $p(x)$ iff $\frac{1}{r}$ is a root of $p^*(x)$

Suppose, for some degree-n polynomial $p(x)$, we define

$$p^*(x) = x^n p\left(\frac{1}{x}\right)$$

What are some properties of $p^*(x)$ we can identify?

1) Suppose r is a root of $p(x)$, assume $r \neq 0$

$$p^*\left(\frac{1}{r}\right) = \left(\frac{1}{r}\right)^n p\left(\frac{1}{\frac{1}{r}}\right) = \left(\frac{1}{r}\right)^n p(r) = 0$$

2) Suppose $\frac{1}{r}$ is a root of $p^*(x)$, assume $r \neq 0$

$$p^*\left(\frac{1}{r}\right) = 0$$

$$\left(\frac{1}{r}\right)^n p(r) = 0$$

since $\left(\frac{1}{r}\right)^n \neq 0$, must have $p(r) = 0$, i.e. r is a root of $p(x)$

$$p(x) = ax^2 + bx + c$$

$$p^*(x) = x^2 p\left(\frac{1}{x}\right)$$

$$= x^2\left(a\left(\frac{1}{x}\right)^2 + b\left(\frac{1}{x}\right) + c\right)$$

$$= \cancel{x^2}\cdot\frac{a}{\cancel{x^2}} + \cancel{x^2}^{\,x}\cdot\frac{b}{\cancel{x}} + cx^2$$

$$= cx^2 + bx + a = 0$$

$$f(x) = x^2 - 7x + 6 \qquad r_1 = 1,\ r_2 = 6$$

$$(x-1)\left(x - \frac{1}{6}\right) = 0$$

$$x^2 - \frac{1}{6}x - x + \frac{1}{6} = 0$$

$$x^2 - \frac{7}{6}x + \frac{1}{6} = 0$$

$$6x^2 - 7x + 1 = 0$$

Example: Suppose $x^3 - 3x^2 + 1$ has roots a, b, c.

a) Determine a polynomial with roots $a+3, b+3, c+3$.

b) Determine a polynomial with roots $\frac{1}{a+3}, \frac{1}{b+3}, \frac{1}{c+3}$.

c) Determine $\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3}$.

a) $(x-3)^3 - 3(x-3)^2 + 1$

$= x^3 - 9x^2 + 27x - 27 - 3(x^2 - 6x + 9) + 1$

$= x^3 - 12x^2 + 45x - 53$

b) $[1 \quad -12 \quad 45 \quad -53]$

$[-53 \quad 45 \quad -12 \quad 1]$ $\longrightarrow$ $-53x^3 + 45x^2 - 12x + 1$