Lecture 22: Review

Introduction to Mathematical Thinking

April 23, 2019

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extra credit
assignment

Announcements

- Homework 8, the final homework for everyone, is now out
    - Due Sunday at 11:59PM
- Final feedback form will be created by Thursday – please fill it out once it is ready!
- **Quiz 5 a week from today**
    - Will only be on topics since Quiz 4

Today and Thursday: Review and doing problems from earlier in the semester.

Next week:

- Tuesday: Quiz only, class will be done early
- Thursday: Wrap-up lecture. Will talk about course advice and look at a cool application of some of the stuff we've seen.

Recap: Binomial Theorem

The binomial theorem states

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$= \binom{n}{0} x^k + \binom{n}{1} x^{k-1} y + \binom{n}{2} x^{k-2} y^2 + \ldots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

We define the k-th term, i.e. the **general term**, in the expansion of a binomial as

$$t_k = \binom{n}{k} x^{n-k} y^k$$

with $k \in \{0, 1, 2, \ldots, n\}$.

Recap: Vieta's Formulas

Degree-2 case: $p(x) = a_2x^2 + a_1x + a_0$

$$r_1 + r_2 = -\frac{a_1}{a_2} \qquad r_1r_2 = \frac{a_0}{a_2}$$

Degree-3 case: $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$

$$r_1 + r_2 + r_3 = -\frac{a_2}{a_3} \qquad r_1r_2 + r_1r_3 + r_2r_3 = \frac{a_1}{a_3} \qquad r_1r_2r_3 = -\frac{a_0}{a_3}$$

Generalize?

Generalized Vieta's Formulas

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

$$= a_n \sum_{k=0}^{n} (-1)^k \left(\text{sum of the products of the roots of } p(x)\text{, taken } k \text{ at a time}\right) x^{n-k}$$

Sum of roots:

$$r_1 + r_2 + \ldots + r_n = -\frac{a_{n-1}}{a_n}$$

Product of roots:

$$r_1 \cdot r_2 \cdot \ldots \cdot r_n = (-1)^n \frac{a_0}{a_n}$$

Overview

$|A \vee B| = |A| + |B| - |A \wedge B|$

1. Sets, Functions, and Logic

- Sets
  - Various set operations – union, intersection, difference, product
  - Principle of Inclusion-Exclusion
  - Definitions of $\underbrace{\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}}$, $\mathbb{R}$, $\mathbb{C}$
    - Relative cardinalities – countable vs. uncountable infinity

countable

- Functions

$D \times CD$

  - Injections, surjections, bijections
  - Domain, codomain, range

$A \Rightarrow B \equiv \neg A \vee B$

- Propositional Logic
  - Basic operators – $\vee$, $\wedge$, $\neg$
  - Implications, and their variants (contraposition, converse)
  - Truth tables

$A \Rightarrow B$

contrap. $\neg B \Rightarrow \neg A$

conv. $B \Rightarrow A$

2. Proof Techniques

- Direct Proofs
- Proof by Contradiction
- Proof by Contraposition
- Proof by Cases
- Proof by Induction
    - Strong induction

prove $A \Rightarrow B$

$\neg B \Rightarrow \neg A$

prove statement about $\mathbb{N}$

Also talked about series and sequences.

telescoping sums

3. Number Theory

- Prime factorization
- Fundamental Theorem of Arithmetic, Division Algorithm
- Modular Arithmetic
    - Finding inverses

$n = dq + r$

$\binom{n}{k}$

4. Counting

- Permutations and combinations - order matters vs. order doesn't matter
- Stars and bars counting
- Pascal's Triangle
- Combinatorial proofs

5. Combinatorics with Polynomials

- Binomial Theorem
- Vieta's Formulas

function

Example: Consider the mapping from $\mathbb{R}$ to $\mathbb{R}$ given by $f : x \mapsto 5x^3 - 2x^2 + 3$. Is f...

a) An injection?

b) A surjection?

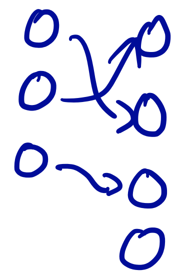
c) A bijection?

$f: A \to B$

Injection

$f(a) = f(b) \Rightarrow a = b$

$a \neq b \Rightarrow f(a) \neq f(b)$



$|A| \leq |B|$

required for injection to exist

a) $f'(x) = 15x^2 - 4x$

$f(x) = 5x^3 - 2x^2 + 3 = 3$

$\rightarrow 5x^3 - 2x^2 = 0$

$x^2(5x - 2) = 0$

$\Rightarrow x = 0$ and $x = \frac{2}{5}$

both set $f(x) = 3$

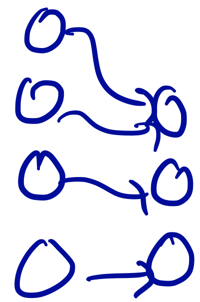
but $0 \neq 2/5$

$\therefore$ not injection!

Surjection

codomain = range

$\forall b \in B, \exists a \in A : f(a) = b$



$|A| \geq |B|$

b) $\boxed{\text{yes}}$

c) no, bc not injection

Example: Prove that there are no integer solutions to $x^2 - 3 = 4y$.

Proof by contradiction

Assume x, y integers satisfying $x^2 - 3 = 4y$

Case 1 x is even

$x = 2k, \quad k \in \mathbb{Z}$

$(2k)^2 - 3 = 4y$

$\underbrace{\underbrace{4k^2}_{\text{even}} - \underbrace{3}_{\text{odd}}}_{\text{odd}} = \underbrace{4y}_{\text{even}}$

odd $\neq$ even, contradiction!

Case 2 x is odd

$x = 2k+1, \quad k \in \mathbb{Z}$

$(2k+1)^2 - 3 = 4y$

$4k^2 + 4k + 1 - 3 = 4y$

$4k^2 + 4k - 2 = 4y$

$\underbrace{2k^2 + 2k - 1}_{\text{odd}} = \underbrace{2y}_{\text{even}}$

Contradiction!

$\therefore$ no int solutions

Example: Prove, using induction, that

$$\sum_{i=0}^{n} 2^{-i} = 2 - \cancel{2^n}\; 2^{-n}$$

Base Case

$n = 0$

LS	RS
$\sum_{i=0}^{0} 2^{-i} = 2^{-0}$	$2 - 2^{-0}$
	$= 2 - 1$
$= 1$ ✓	$= 1$

IH

Assume $\sum_{i=0}^{k} 2^{-i} = 2 - 2^{-k}$

Induction Step

induction hyp.

$$\sum_{i=0}^{k+1} 2^{-i} = \sum_{i=0}^{k} 2^{-i} + 2^{-(k+1)}$$

$$= 2 - 2^{-k} + 2^{-(k+1)}$$

$$= 2 - \frac{2}{2^k \cdot 2} + \frac{1}{2^{k+1}}$$

$$= 2 - \frac{1}{2^{k+1}}$$

$$= 2 - 2^{-(k+1)}$$ ✓

Example: The harmonic series $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$ is known to be unbounded as $n \to \infty$.

1) Using induction, prove that $\forall n \in \mathbb{N}$, $H_{2^n} \geq 1 + \frac{n}{2}$.
2) *Why does this prove that $H_n \to \infty$ as $n \to \infty$?*

Base Case $n = 0$

$H_{2^0} = H_1 = 1 \Rightarrow 1 \geq 1$, ∴ BC holds

$1 + \frac{0}{2} = 1$

IH

Assume

$$\boxed{H_{2^k} \geq 1 + \frac{k}{2}}$$

IS

H_{2^k}

$$H_{2^{k+1}} = \underbrace{1 + \frac{1}{2} + \ldots + \frac{1}{2^k}}_{2^k} + \underbrace{\frac{1}{2^k+1} + \frac{1}{2^k+2} + \ldots + \frac{1}{2^{k+1}}}_{2^k}$$

$$\geq H_{2^k} + 2^k \cdot \frac{1}{2^{k+1}} \quad \frac{1}{2}$$

$$\geq 1 + \frac{k}{2} + \frac{1}{2} = \boxed{1 + \frac{k+1}{2}}$$

Example: Prove

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}} = 2$$

$$x = \sqrt{2}^{\,x}$$

$$\log x = x \log 2^{1/2}$$

$$\log x = \frac{1}{2} x \log 2$$

$$\frac{\log x}{x} = \frac{\log 2}{2}$$

$x = 2$ and $x = 4$

both satisfy!

$t_0 = \sqrt{2}$

$t_n = \sqrt{2}^{\,t_{n-1}}$

Prove $t_n \le 2$

BC

$t_0 = \sqrt{2} \le 2$

Example: Prove that if p is a prime, $p \geq 5$, then $p = 6k + 1$ or $p = 6k - 1$ for some $k \in \mathbb{N}$, using

a) A direct proof

b) A proof by contraposition

a) $p-1$ $\quad p \quad$ $p+1$

even $\quad\quad$ even

in any 3 consecutive ints, 1 is a multiple of 3

1) $p+1$ is mult of 3

$\Rightarrow$ $p+1$ is mult of 6

$\Rightarrow$ $p+1 = 6k$

$\Rightarrow$ $\boxed{p = 6k-1}$

2) $p-1$ is mult of 3

$\Rightarrow$ $p-1 \equiv 0 \mod 6$

$\Rightarrow$ $p-1 = 6k$

$\Rightarrow$ $\boxed{p = 6k+1}$

b) p prime $(p \geq 5) \implies p = 6k+1$ or $p = 6k-1$

$\rightarrow$ if $p \neq 6k+1$ and $p \neq 6k-1$ then p not prime

$p = 6k \rightarrow$ mult of 6, not prime

$p = 6k+2 = 2(3k+1) \rightarrow$ not prime

$p = 6k+3 = 3(2k+1) \rightarrow$ not prime

$p = 6k+4 = 2(3k+2) \rightarrow$ not prime

$(-c)^n = -c^n$ n odd

$(-c)^n = c^n$ n even

Example: Suppose n is an odd positive integer, and suppose we have

$$L(n) = \text{the last digit of } n$$

$L(n) \equiv n \mod 10$

Determine $L(\sum_{k=1}^{10} k^n)$.

$$\sum_{k=1}^{10} k^n = 1^n + 2^n + \cdots + 10^n$$

$9 \equiv -1$

$8 \equiv -2$

$7 \equiv -3$

$6 \equiv -4$

$\mod 10$

$$L\left(\sum k^n\right) = 1^n + 2^n + 3^n + 4^n + 5^n + 10^n + 9^n + 8^n + 7^n + 6^n \mod 10$$

$$\equiv 1^n + 2^n + 3^n + 4^n + 5^n + 10^n + (-1)^n + (-2)^n + (-3)^n + (-4)^n$$

$$\equiv \cancel{(1^n - 1^n)} + \cancel{(2^n - 2^n)} + \cancel{(3^n - 3^n)} + \cancel{(4^n - 4^n)} + 5^n + \cancel{10^n} \mod 10$$

$$\equiv \boxed{5}$$