Lecture 23: Review

Introduction to Mathematical Thinking

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Announcements

- **Quiz 5 on Tuesday**
  - This week and next week still count for attendance
- Homework 8 due Sunday 11:59pm
- Final feedback form released: please fill it out!

extra credit released after quiz

Today: More review.

Next week:

- Tuesday: Quiz only, class will be done early
- Thursday: Wrap-up lecture. Will talk about course advice and look at a cool application of some of the stuff we've seen.

$Z_3 = \{0, 1, 2\}$

Example: In the following, Z_n refers to the set of integers modulo n.

a) Is $f : x \mapsto 7x$ a bijection from Z_{12} to Z_{12}? $\gcd(7, 12) = 1$

b) Is $f : x \mapsto 6x$ a bijection from Z_{12} to Z_{12}? $\gcd(6, 12) \neq 1$

c) Does there exist an injection from Z_{12} to Z_{24}? A surjection? In both cases, if so, identify them.

a) yes

7 has an inverse in mod 12

injection $f(a) = f(b) \Rightarrow a = b$

$7a \equiv 7b$

$7^{-1} 7a \equiv 7^{-1} 7b$

$a \equiv b$ ✓

surjection suppose c is the output of f. $f(x) = c \Rightarrow 7x = c$

$x = 7^{-1} c$ ✓

b) no

$f(2) \equiv 0$

$f(4) \equiv 0$

$2 \neq 4$

$\therefore$ not inj

$\therefore$ not bij.

$A : \mathbb{Z}_{12}$

$B : \mathbb{Z}_{24}$

injection

$f : x \mapsto x$

$|A| \le |B|$

surjection

$|A| \ge |B|$

$\therefore$ no

Example: Suppose A, B are two sets.

a) Suppose $|A| = |B| = n$. How many bijections are there from A to B?

b) Suppose $|A| = a$ and $|B| = b$. How many functions are there from A to B?

c) Suppose again that $|A| = a$ and $|B| = b$, and $a < b$. How many injections are there from A to B?

a)

$a_1 \quad b_1$

$a_2 \quad b_2$

$a_3 \quad b_3$

$\vdots \quad \vdots$

$a_n \quad b_n$

$$n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1$$

$$= n!$$

b)

$x_1 \quad y_1$

$x_2 \quad y_2$

$\vdots \quad \vdots$

$x_a \quad y_b$

$$b \cdot b \cdot \ldots \cdot b = b^a$$

c)

$$b(b-1)(b-2)\ldots(b-a+1)$$

$$= \frac{b!}{(b-a)!} = P(b,a)$$

Example: Give a combinatorial proof of the following identity, assuming $n \geq r \geq k$:

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

Choosing a team of r players
k being starters

LS: 1) choose the team $\binom{n}{r} \cdot \binom{r}{k}$
2) choose the starters

RS: 1) choose the starters $\binom{n}{k} \cdot \binom{n-k}{r-k}$

2) choose rest of team

Example: Prove the power rule of derivatives, using the Binomial Theorem. That is, prove

$$\lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

for $n \geq 1$.

Hint: You will need the limit definition of derivatives.

$$\frac{df}{dx}(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$(x+h)^n = \cancel{\binom{n}{0}x^n} + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n$$

$$-x^n$$

$$\frac{(x+h)^n - x^n}{h} = \binom{n}{1}x^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + \binom{n}{n-1}xh^{n-2} + h^{n-1}$$

$$\lim_{h\to 0}\left(\quad\right) = \binom{n}{1}x^{n-1} = nx^{n-1}$$

Example: In the expansion of

$c \in \mathbb{R}$

$$\left(2x - \frac{c}{x}\right)^{10}$$

10^5

the value of the constant term (i.e. the term independent of x) is 100000. Determine c.

$$t_k = \binom{10}{k}(2x)^{10-k}\left(-cx^{-1}\right)^k$$

$$(x+y)^{10} = \sum_{k=0}^{10}\binom{10}{k}x^{10-k}y^k$$

$$= (-1)^k\binom{10}{k}2^{10-k}c^k x^{10-2k}$$

$$10-2k=0 \Rightarrow k=5$$

$$\text{coef}(t_5) = (-1)^5\binom{10}{5}\cancel{2^5}c^5 = \cancel{10^5}\ 5^5$$

$$\Rightarrow c = \left(\frac{-5^5}{\binom{10}{5}}\right)^{1/5}$$

Example: What is the coefficient of the $x^2y^2z^2$ term in the expansion of $(x+y+z)^6$?

$0 \le a, b, c \le n$

General: $\frac{n!}{a!b!c!} x^a y^b z^c$ $\quad a+b+c=n$

$x^{2} y^{-} z^{-}$

In our case

$\frac{6!}{2!2!2!} x^2 y^2 z^2$

$\Uparrow$

Example: If the sum of the reciprocals of the roots of the quadratic

$$3x^2 + 7x + k = 0$$

is $\frac{7}{3}$, what is k?

$$r_1 + r_2 = -\frac{7}{3}$$

$$r_1 r_2 = \frac{k}{3}$$

sum of reciprocals

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1 + r_2}{r_1 r_2} = \frac{7}{3}$$

$$\frac{-\frac{7}{3}}{\frac{k}{3}} = \frac{7}{3}$$

$$\Longrightarrow \boxed{k = -3}$$

Example: Let $f(x) = x^4 - 14x^3 + 15x - 13$ have roots r_1, r_2, r_3 r_4. Find

$$r_1^2 r_2 r_3 r_4 + r_1 r_2^2 r_3 r_4 + r_1 r_2 r_3^2 r_4 + r_1 r_2 r_3 r_4^2$$

$$= r_1 r_2 r_3 r_4 (r_1 + r_2 + r_3 + r_4)$$

$$= \boxed{-13 \cdot 14} = -182$$