Lecture 24: Probability, Closing Thoughts

Introduction to Mathematical Thinking

May 2, 2019

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1) don't have pts
2) more than 3 unexcused

Announcements

- Grading breakdown will be released tomorrow
    - You will receive an email saying whether or not you've passed, and whether or not you need the extra credit assignment to pass
    - If the email you receive says you've already passed, then you're good to go!
    - The extra credit assignment will be released alongside it, and will be due on the **Friday of dead week**
- Feedback form **due today**
- Today: A new topic that builds upon what we've seen...

Introduction to Probability

We deliberately didn't touch upon probability this semester. With that being said, many of the ideas in probability build upon the content we've seen so far this semester.

$\Omega = \{1, 2, 3, 4, 5, 6\}$ dice

$\Omega = \{2\heartsuit, 3\diamondsuit, \ldots\}$

omega

Probability Axioms

Let Ω represent our "event space" – the set of all possible "outcomes" of our event, and suppose $A, B \subseteq \Omega$.

e.g. $A = \{2, 4, 6\}$

$B = \{4, 5, 6\}$

axioms

1. $\mathbb{P}(A) \geq 0, \forall A \subset \Omega$
2. $\mathbb{P}(\Omega) = \sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$
3. $\text{If } A, B \text{ disjoint, then } \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$

$|A \cup B| = |A| + |B|$ iff A, B disjoint

$A \cap B = \{\} = \emptyset$

$|A \cap B| = 0$

At it's core:

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|}$$

$$P(B) = \frac{|\{4, 5, 6\}|}{|\{1, 2, 3, \ldots 6\}|} = \frac{3}{6} = \frac{1}{2}$$

Some Definitions...

- We say two events are **mutually exclusive** iff $\mathbb{P}(A \cap B) = 0$, i.e. iff they are disjoint
  - For example: When drawing a single card from a deck of cards, one cannot draw both a black card and a red card – these two sets are disjoint. Therefore, $\mathbb{P}(\text{card is black} \cap \text{card is red}) = 0$.
  - However, one could draw both a black card and a Queen, and so $\mathbb{P}(\text{card is black} \cap \text{card is Queen}) > 0$.
- We say two events are **independent** iff

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

<u>independent</u>

$P(A \cap B) = P(A)\,P(B)$

<u>mutually exclusive</u>

$P(A \cap B) = 0$

$P(A)\,P(B) = 0$

$$P(A^c) = 1 - P(A)$$
$$\Rightarrow P(A) + P(A^c) = 1$$

Counting Problems Return!

Before: How many ways can we arrange the characters in BERKELEY, such that BERK appears as a substring?

$\boxed{BERK}$
EE
L
Y

$$\frac{5!}{2!}$$

Total
B
EEE
R
K
L
Y

$$\frac{8!}{3!}$$

Now: What is the *probability* that a random permutation of BERKELEY contains BERK as a substring?

$$\text{prob} = \frac{\frac{5!}{2!}}{\frac{8!}{3!}}$$

$$\text{prob not BERK} = \frac{\text{total} - \#\text{ ways BERK}}{\text{total}}$$

$$= 1 - \text{prob BERK}$$

Example: Suppose we have a group of 8 juniors and 5 seniors. What's the probability that a randomly selected group of 4 of these students contains 2 juniors and 2 seniors?

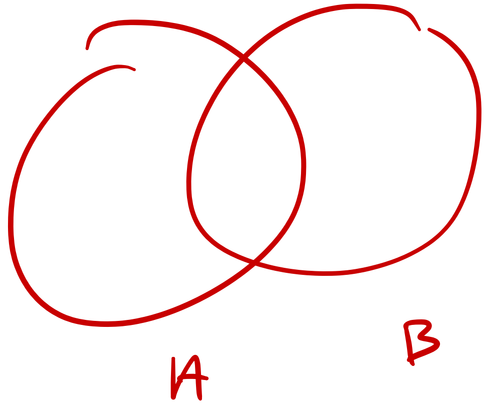
$$P(2J, 2S) = \frac{\binom{8}{2}\binom{5}{2}}{\binom{13}{4}}$$

at least 2 juniors:

$$P(2J, 2S) + P(3J, 1S) + P(4J, 0S)$$

$$= \frac{\binom{8}{2}\binom{5}{2} + \binom{8}{3}\binom{5}{1} + \binom{8}{4}\binom{5}{0}}{\binom{13}{4}}$$

Principle of Inclusion-Exclusion



Recall, for two (finite) sets A, B:

$$P\ |A \cup B| = P\,|A| + P\,|B| - P\,|A \cap B|$$

We can apply this idea to the probability as well. Suppose $A, B \subseteq \Omega$. Then,

$$\mathbb{P}(A \cup B) = \frac{|A \cup B|}{|\Omega|}$$

$$= \frac{|A| + |B| - |A \cap B|}{|\Omega|}$$

$$= \frac{|A|}{|\Omega|} + \frac{|B|}{|\Omega|} - \frac{|A \cap B|}{|\Omega|}$$

$$\Rightarrow \mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

Example: What is the probability that a randomly selected number in the range $[1, 1000]$ (inclusive) is a multiple of 2 or 3?

$$p(\text{mult } 2 \cup \text{mult } 3) = \frac{\#\text{Mult } 2 + \#\text{Mult } 3 - \#\text{Mult } 6}{1000}$$

$$= \frac{500 + 333 - 166}{1000}$$

Random Variables

A *random variable* is a function that maps outcomes of a random process to real numbers.

With (discrete) random variable X, we associate a **probability mass function:**

$$\overset{\text{p.m.f.}}{f_X}(x) = \mathbb{P}(\overset{\text{r.v.}}{X} = \overset{\text{value}}{x})$$

This denotes the probability that random variable X assumes the value x.

Note:

- Discrete random variables: set of outcomes Ω is countable (finite or countably infinite)
- Continuous random variables: set of outcomes Ω is uncountable

For example, suppose X represents the outcome of the roll of a dice.

Then, $\Omega = \{1, 2, 3, 4, 5, 6\}$, and

$$\mathbb{P}(X = x) = \frac{1}{6}$$

since all outcomes on a dice are equally likely.

$P(X=1)$
$= P(X=2)$
$= \dots = P(X=6)$
$= \boxed{\frac{1}{6}}$

Let's look at a few examples of more involved random variables.

Example: Number of Failures

Suppose the probability of me making a free throw is $p = 0.6$, and suppose each free throw is independent. What's the probability that my first made shot is on my 8th shot (trial)?

8 trials: 7 failures, then 1 success $\quad 0.4 \cdot 0.4 \cdot 0.4 \cdots\cdots 0.4 \cdot 0.6$

$$\mathbb{P}(7 \text{ misses, then } 1 \text{ make}) = \mathbb{P}(\text{miss 1st}) \cdot \mathbb{P}(\text{miss 2nd}) \cdot \ldots \cdot \mathbb{P}(\text{miss 7th}) \cdot \mathbb{P}(\text{make 8th})$$

$$= 0.4^7 \cdot 0.6$$

$$= (1-p)^7 p$$

$$P(\text{first make is shot } 100) = 0.4^{99}\, 0.6$$

Geometric Distribution

Suppose an event succeeds with probability p, and each trial of the event is independent. Let X be the random variable that describes the number of trials required until the first success. Then, the probability mass function of X is given by

$$\mathbb{P}(X = k) = (1 - p)^{k-1} p$$

prob. that trial k is the first success

outcomes : infinitely many

$\Rightarrow$ still discrete!

Example: Number of Successes in n trials

Now, suppose some event succeeds with probability p, and we repeat it n times, and want to model the probability of k successes.

For example, let's say we have a coin that flips heads with probability $p = 0.6$. Suppose we flip the coin 5 times, and want to find the probability that it lands heads 3 times.

Each of our 5 flips could each be heads or tails, e.g.

$$\text{HHTTT, HTHTH, TTTHT, ...}$$

How many permutations of our flips have exactly three heads?

$$\frac{5!}{3!\,2!} = \binom{5}{3}$$

- Our problem is to find the number of permutations of HHHTT.
- We have 5 flips, and want to choose 3 of them to be heads. We can do this in $\binom{5}{3}$ ways
- Any sequence of 5 flips with 3 heads and 2 tails has probability

$$\underbrace{p^3}_{\text{3 heads}}\overbrace{(1-p)^2}^{\text{2 tails}} = 0.6^{3}0.4^{2}$$

Then...

$$P(X=3) = \binom{5}{3} p^3 (1-p)^2$$

$$= \binom{5}{3} \underbrace{0.6^3\ 0.4^2}$$

Binomial Distribution

independent

Suppose an event succeeds with probability p, and each trial of the event is ~~indepedent~~. Let X be the random variable that describes the number of successes in n trials. Then, the probability of k successes is given by

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Look familiar?

general term of $(p + (1-p))^n$

possible outcomes : $n + 1$

$n > 365: \ p = 1$

The Birthday Paradox

There are 365 unique days in a year. **What's the probability that in a group of n people, at least two people share a birthday?**

Key insight: There are two disjoint cases that partition our event space.

1. Everyone has a unique birthday
2. At least two people share a birthday

$$\mathbb{P}(\text{everyone has a unique birthday}) + \mathbb{P}(\text{at least two people share a birthday}) = 1$$

$$\Rightarrow \mathbb{P}\underbrace{(\text{at least two people share a birthday})}_{p(n)} = 1 - \mathbb{P}(\text{everyone has a unique birthday})$$

$$p(n) = 1 - \mathbb{P}(\text{all unique}, N = n)$$

Let's consider the 2-person case. What's the probability that in a group of two people, at least two share a birthday?

$$\Rightarrow P(\text{all unique}, N = 2)$$

$$= \frac{365}{365} \cdot \frac{364}{365}$$

$$\Rightarrow p(2) = 1 - \frac{365}{365} \cdot \frac{364}{365}$$

$$P(\text{all unique}, N=2) = \frac{365}{365} \cdot \frac{364}{365}$$

In general...

$$\mathbb{P}(\text{all unique}, N = n) = \frac{365}{365} \cdot \frac{364}{365} \cdot \ldots \cdot \frac{365-(n-1)}{365}$$

$$P(n, k) = \frac{n!}{(n-k)!} = \binom{n}{k} k!$$

$$= \frac{P(365, n)}{365^n}$$

$$= \frac{\binom{365}{n} n!}{365^n}$$

Now, the probability that at least two people share a birthday in a group of size n is the complement of this:

$$\boxed{p(n) = 1 - \frac{\binom{365}{n} n!}{365^n}}$$

Let's look at a plot of n vs. $p(n)$.

Final Thoughts