Lecture 3: Bijections and Number Sets

Introduction to Mathematical Thinking

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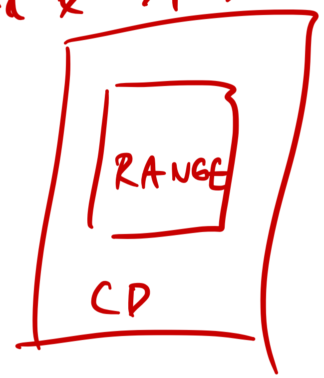
$A - B = \{x : x \in A \text{ and } x \notin B\}$

$B - A = \{x : x \in B \text{ and } x \notin A\}$

In-lecture quiz from last week

What's the difference between the codomain and range of a function? *



- Codomain represents the set of possible inputs, while range represents the set of possible outputs
- These terms mean the same thing
- Codomain represents the set of possible outputs, while range represents the set of actual outputs
- Codomain represents the set of actual outputs, while range represents the set of possible outputs

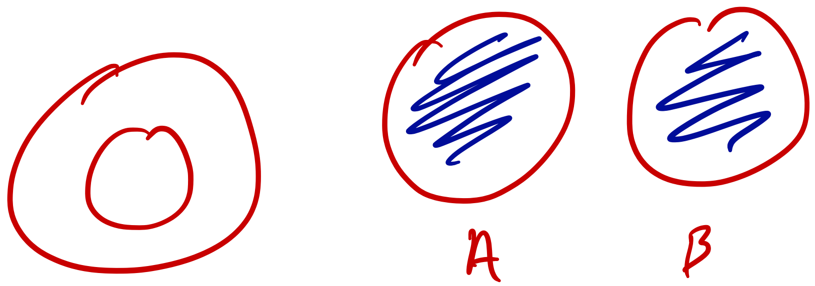
If A and B are two disjoint sets, which of the following are true? Select all that apply. *

$A - B = \{x \in A \text{ and } x \notin B\}$

$|A \cap B| = 0$

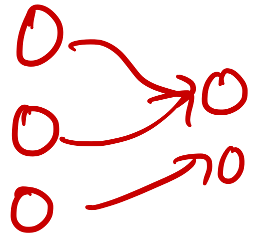
$A \cap B = \emptyset = \{\}$

- The cardinality of the intersection of A and B is 0
- The cardinality of the union of A and B is the sum of the cardinalities of A and B
- The set difference between A and B is equal to just A
- ~~A can be a subset of B, or vice versa~~



PIE: $|A \cup B| = |A| + |B| - \cancel{|A \cap B|}$

Injections



not an injection!

We say a function $f : A \to B$ is **injective**, or **one-to-one**, if no two elements in the input have the same output.

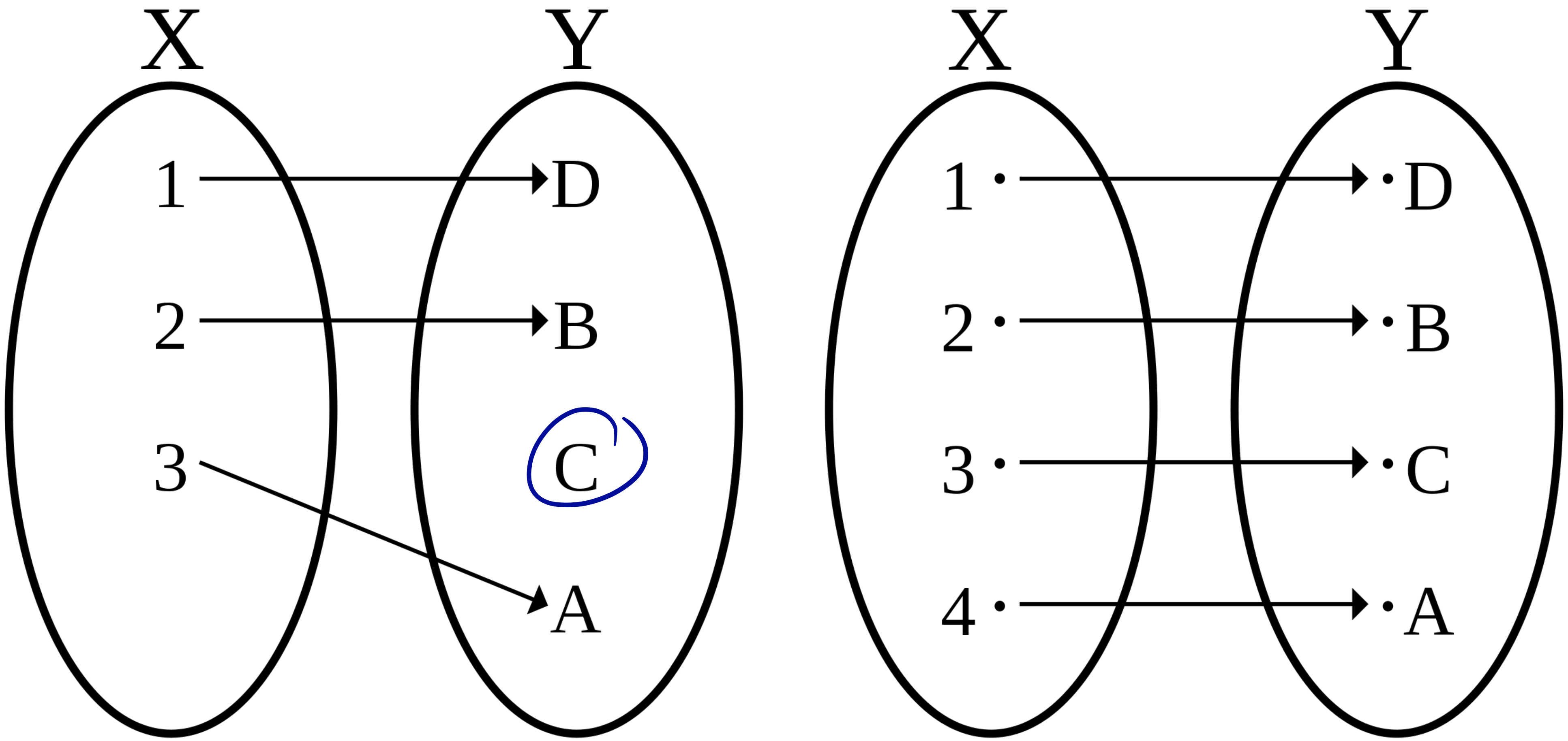
In formal mathematical notation, we'd say

implies

$$\forall x, y \in A, f(x) = f(y) \Rightarrow x = y$$

This translates to "for all x, y in the domain, if x's output is equal to y's output, then x and y must be the same element.

$$\forall x, y \in A \quad x \neq y \implies f(x) \neq f(y)$$



Both of these functions are examples of injections.

$g : \mathbb{R} \to \mathbb{R}$ $\quad g : x \longmapsto \sqrt{x^2}$

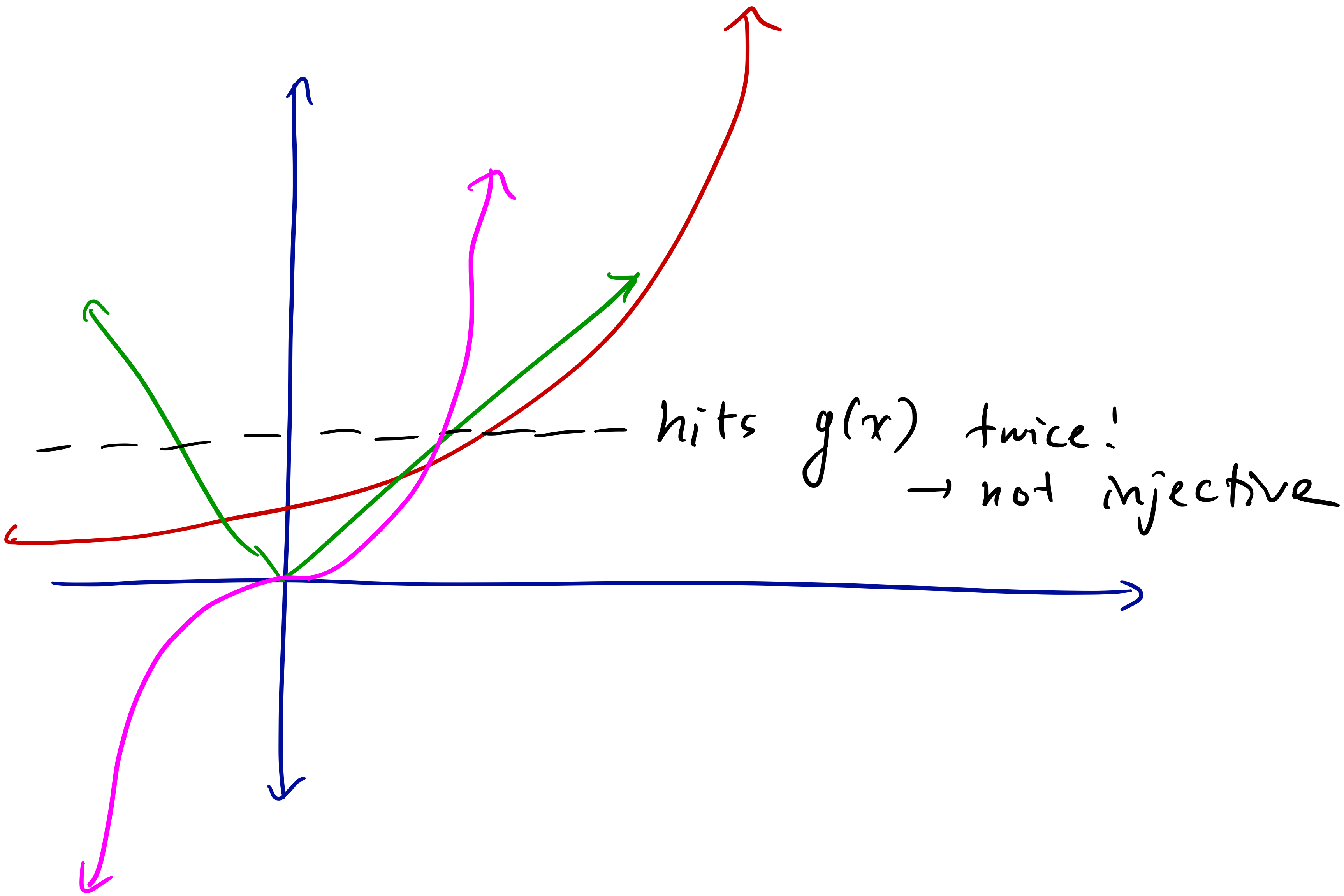
Are the following functions injections?

Your answer may change depending on the domain and codomain!

- $f(x) = e^x$ yes
- $g : x \mapsto \sqrt{x^2}$ no
- $h(x) = x^3 + 3x$ yes

$\sqrt{x^2} = |x|$

$g(-3)$



The "Injection" Test

We know that the derivative $f'(x)$ of a function $f(x)$ describes the rate of change of f. Specifically, $f'(a)$ tells us the rate at which $f(x)$ is changing at $x = a$.

It turns out, if a function's derivative is always greater than 0 or always less than 0, then the function is injective.

$h(x) = x^3 + 3x$

check if $h'(x) > 0$ or $h'(x) < 0$, $\forall x$

therefore

$h'(x) = 3x^2 + 3 \geq 3 > 0$

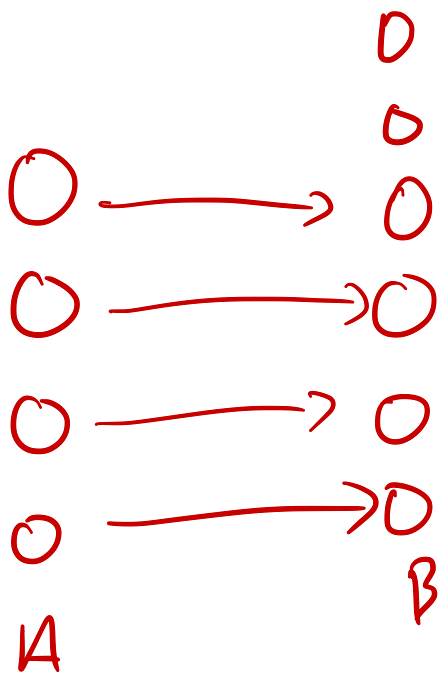
$\therefore$ $h'(x)$ is always > 0

$\therefore$ $h(x)$ is strictly increasing $\rightarrow$ $\boxed{\text{injection!}}$

Injections and Cardinality

Suppose that $h : A \to B$ is an injection. What is the relationship between $|A|$ and $|B|$?

$$|A| \leq |B|$$



need at least as many elements in B as there are in A

$$f(\square) = f(\Delta) \Longrightarrow \square = \Delta$$

injection

Sample Problem

Prove that the *composition* of two injective functions is also injective. In other words, if $f(x)$ and $g(x)$ are both injective, prove that $f(g(x))$ is also injective.

Want to show $f(g(x_1)) = f(g(x_2)) \Longrightarrow x_1 = x_2$

$$f(\boxed{g(x_1)}) = f(g(x_2))$$

since $f(\cdot)$ is injective, this means

$$\boxed{g(x_1)} = g(x_2)$$

since $g(\cdot)$ is injective, this means

$$x_1 = x_2$$

$\therefore$ $f(g(x))$ is injective.

$$f(g(x_1)) = f(g(x_2)) \Longrightarrow g(x_1) = g(x_2) \Longrightarrow x_1 = x_2$$

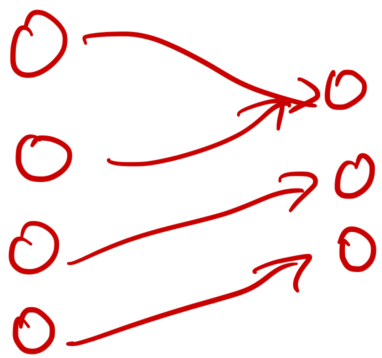
Surjections

We say a function $f : A \to B$ is surjective, or **onto**, if every element in B is mapped to by an element in A, i.e. when the codomain and range are the same set (that is, all possible outputs are actually seen as outputs).

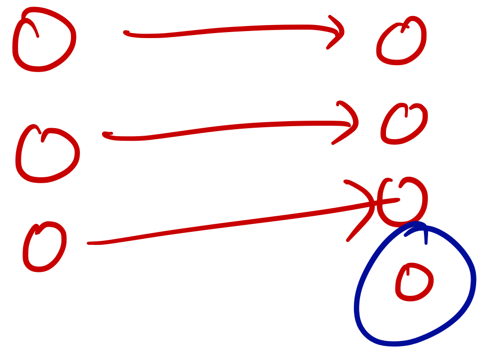
Mathematically, we say

$$\forall\, b \in B, \exists\, a \in A : f(a) = b$$

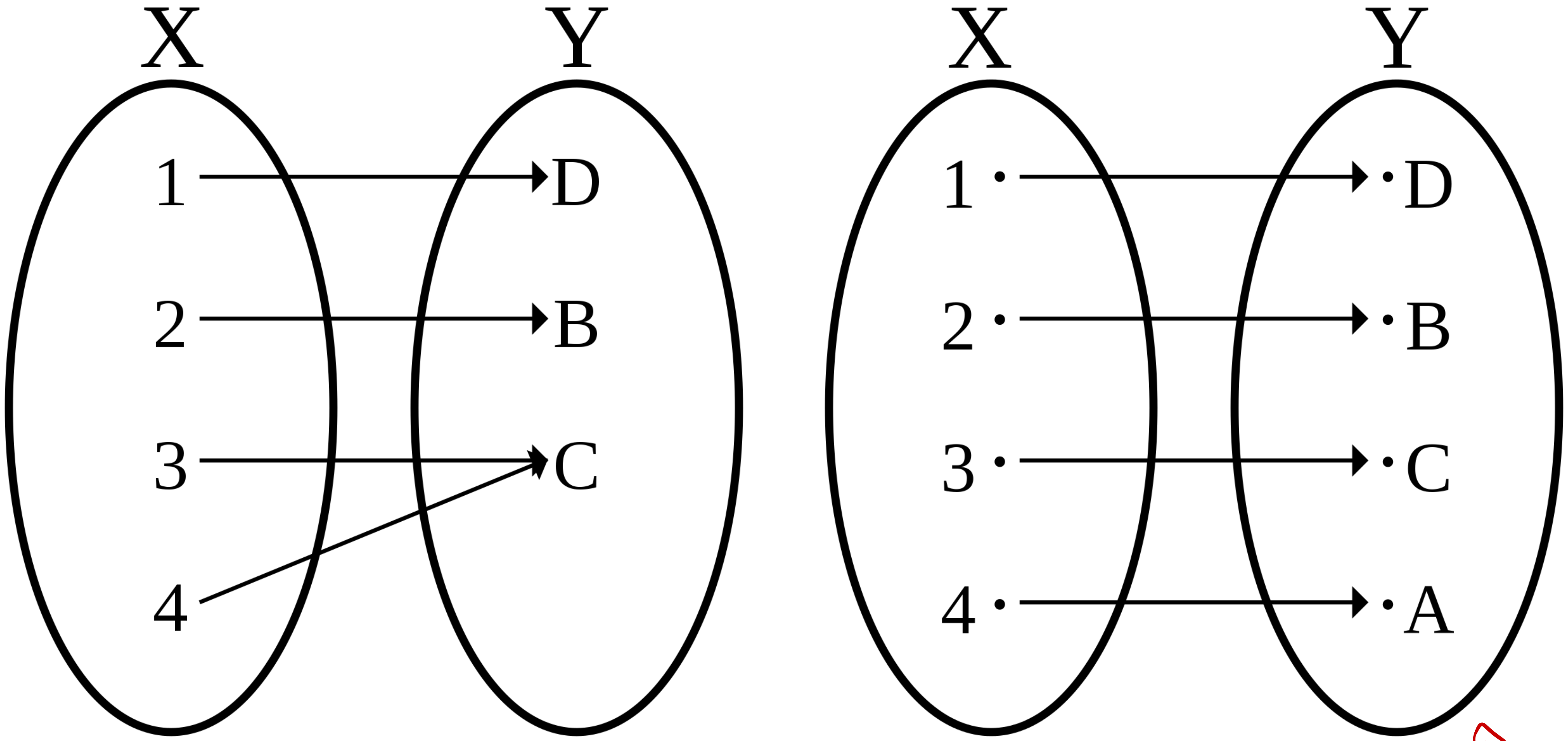
This is read, "for all b in the codomain, there exists some a in the domain such that f maps a to b.



surjection,
but not injection



injection,
but not surjection



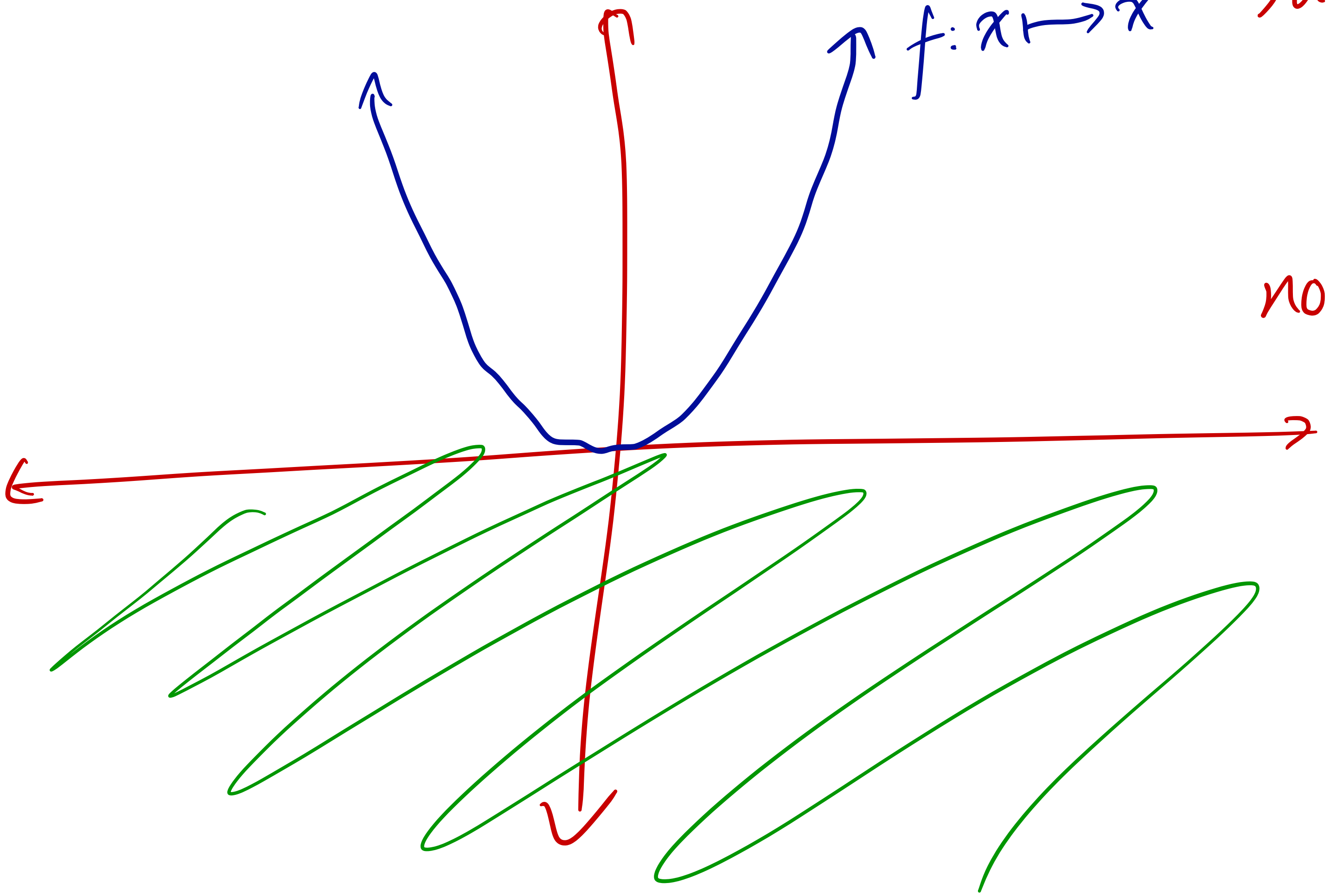
also injection!

Both of these functions are examples of surjections.

Anything look familiar?

What domain and codomain make $f : x \mapsto x^2$ a surjection? Not a surjection?

non-negative



$f: x \mapsto x^2$

surjection: $f: \mathbb{R} \to \mathbb{R}_{\geq 0}$

not surjection: $f: \mathbb{R} \to \mathbb{R}$

all reals less than 0

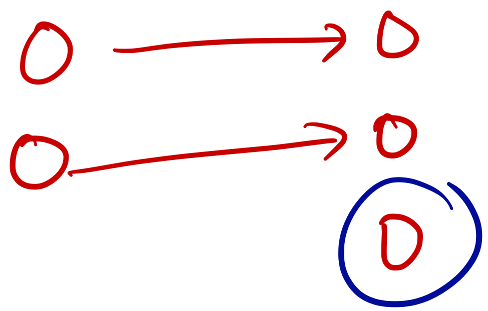
are ignored by function!

Surjections and Cardinality

Suppose that $h : A \to B$ is a surjection. What is the relationship between $|A|$ and $|B|$?

$$|A| \geq |B|$$

need at least as many elements in A as we have in B



P Q

P is true if Q is true

AND

Q is true if P is true

Bijections

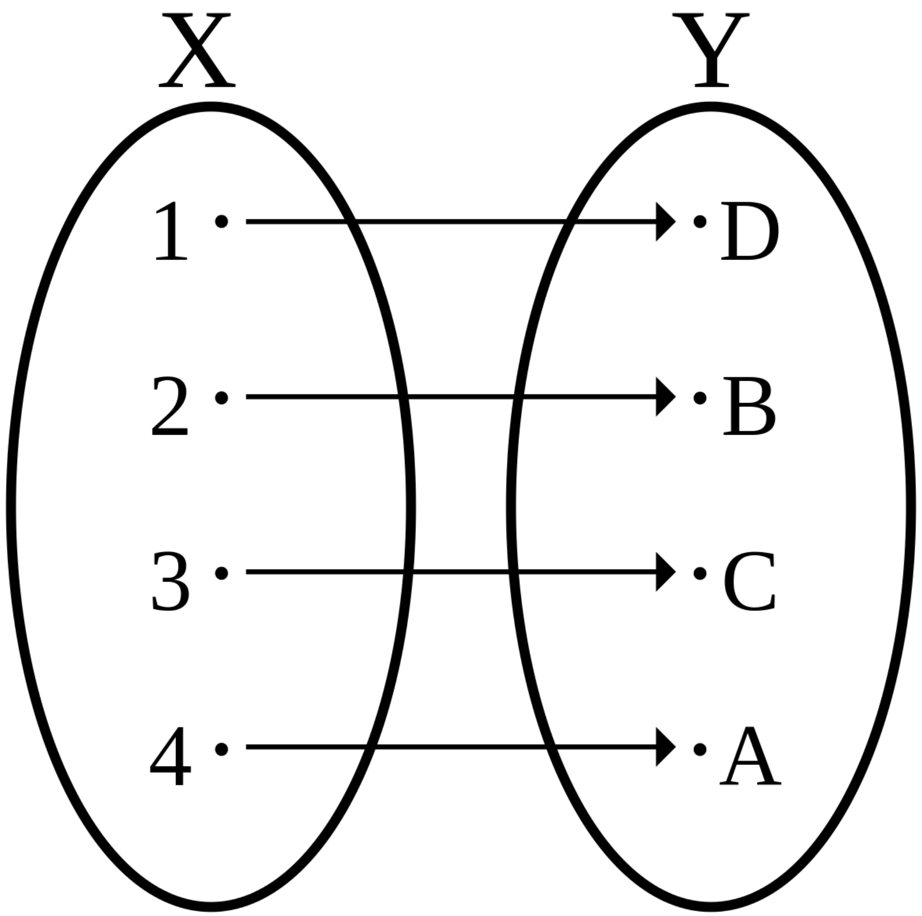
A function f is bijective **if and only if** it is both injective and surjective.

That is, a function is a bijection if and only if:

- no two elements in the domain map to the same element in the range
- every element in the codomain has something mapping to it from the domain.

Additionally: A function is bijective if and only if it has an inverse. That is, $f : A \to B$ is bijective if and only if there exists some $g : B \to A$ such that $\underbrace{f(g(y)) = y}_{\text{right inverse}}$ and $\underbrace{g(f(x)) = x}_{\text{left inverse}}$. Furthermore, this function g is unique, and we often denote it as $f^{-1}(x)$.

$|X| = 4$ $|Y| = 4$



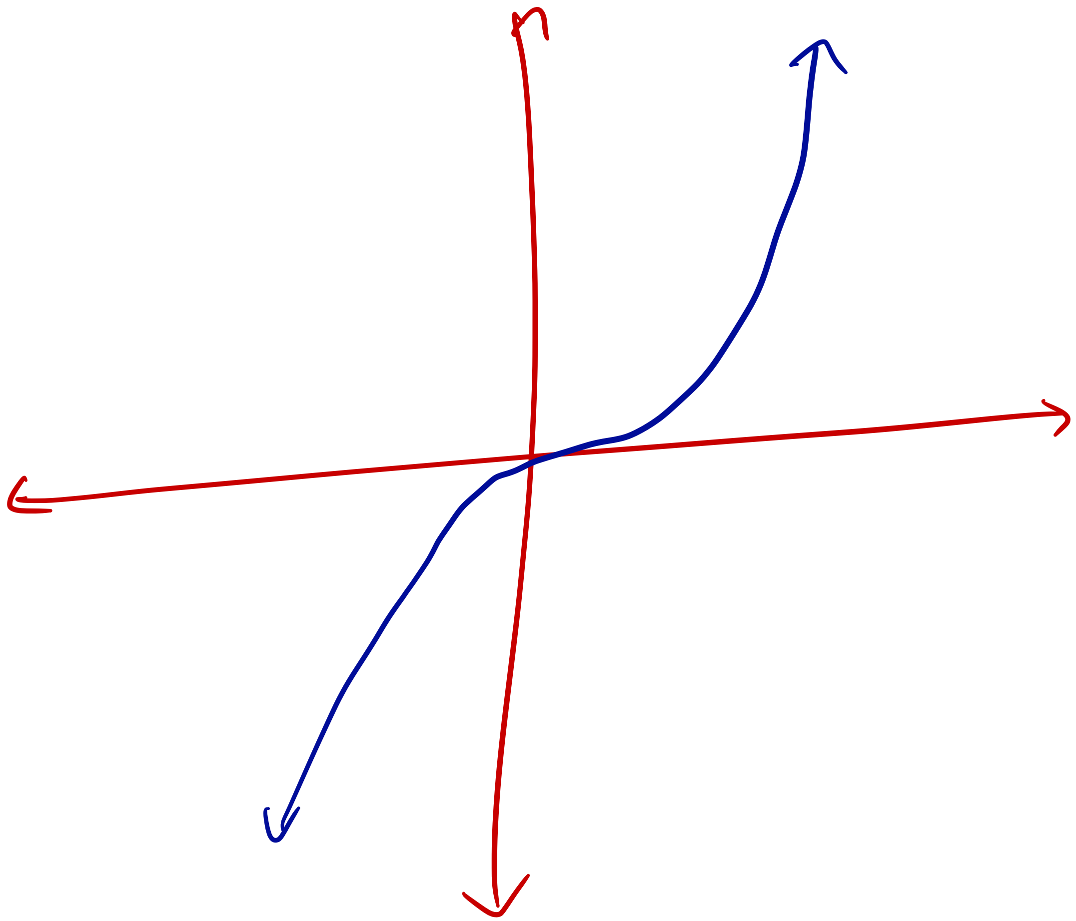
This function is an example of a bijection.

$|X| = |Y|$

Are the following functions bijections?

- $f(x) = x^3$, with domain $\mathbb{R}$ and codomain $\mathbb{R}$
- $d(x, y) = x^2 + y^2$, with domain $\mathbb{R}^2$ and codomain $\mathbb{R}_{\geq 0}$
- $t \mapsto$ the t^{th} prime number, with domain $\{1, 2, 3, 4, \dots\}$ and codomain $\{t : \text{t is prime}\}$

injection ✓
surjection ✓
bijection ✓



injection
$g(3,4) = g(5,0) = 25$
but $(3,4) \neq (5,0)$
NO

surjection
output is $c \in \mathbb{R}_{\geq 0}$
$g(\sqrt{c}, 0) = c$

Bijections and Cardinality

Suppose that $h : A \to B$ is a bijection. What is the relationship between $|A|$ and $|B|$?

injection : $|A| \leq |B|$

surjection : $|A| \geq |B|$

$x \geq y$

$x \leq y$

$\Longrightarrow x = y$

$\Downarrow$

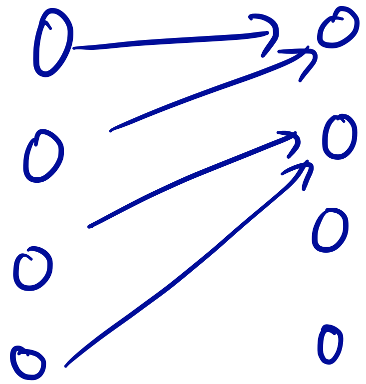
$\boxed{|A| = |B|}$

Another Definition of Cardinality $|A| = |B|$

We can say that two sets A, B have the same cardinality if there exists a bijection $f : A \to B$ between them.

Why does this definition matter?

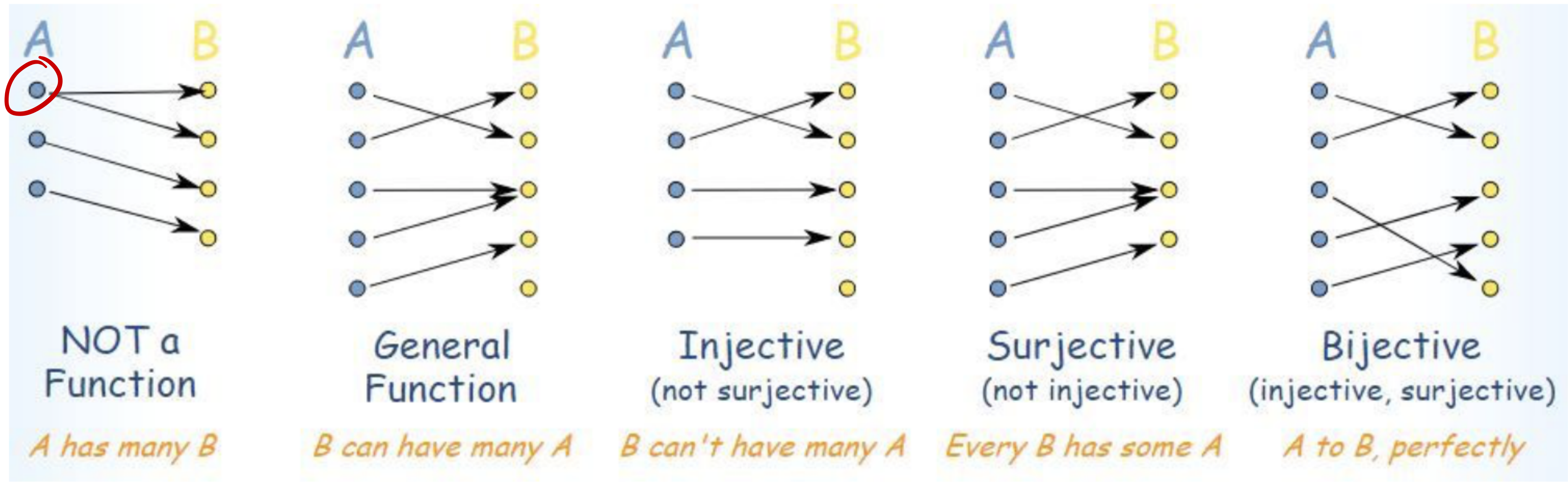
→ extends to infinitely large sets



not every function will be a bijection

→ we can create one

Summary of Types of Relations



- **Function**: For every x, there is exactly one y
- **Injection**: Function where no two xs map to the same y
- **Surjection**: Function where for every y, there is ~~exactly~~ at least one x
- **Bijection**: Both an injection and surjection

Number Sets

Now, we'll formally define and investigate the relationships between the various number sets we take for granted. More specifically, we'll want to determine the **relative sizes** of each of these sets.

- Natural numbers
- Whole numbers
- Integers
- Rational numbers
- Irrational numbers
- Real numbers
- Complex numbers

Our new definition of cardinality will be extremely useful.

Natural Numbers

The **natural numbers** (also known as the counting numbers), denoted by $\mathbb{N}$, are the most primitive numbers; ones that occur trivially in nature that can be used to count a (non-zero) number of things.

$$\mathbb{N} = \{1, 2, 3, 4, ...\}$$

Whole Numbers

The set of **whole numbers**, denoted by $\mathbb{N}_0$, is the union of the set of counting numbers with the number 0.

$$\mathbb{N}_0 = \{0, 1, 2, 3, 4, ...\} = \{0\} \cup \mathbb{N}$$

**Note: In some texts, $\mathbb{N}$ refers to $\{0, 1, 2, ...\}$ as opposed to $\{1, 2, 3, ...\}$. Keep in mind this distinction for our purposes.*

Is there a bijection between the natural numbers and whole numbers?

$\mathbb{N} = \{1, 2, 3, \ldots\}$

$\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$

There is a bijection from $\mathbb{N} \to \mathbb{N}_0$

(also, $\mathbb{N}_0 \to \mathbb{N}$)

$f: \mathbb{N}_0 \to \mathbb{N} \qquad f(n) = n+1$

$\therefore \; |\mathbb{N}| = |\mathbb{N}_0|$

$\therefore$ wholes are countably inf.

$$\begin{array}{lcccccc} \mathbb{N}: & 1, & 2, & 3, & 4, & \ldots \\ & \uparrow & \uparrow & \uparrow & \uparrow & \\ \mathbb{N}_0: & 0, & 1, & 2, & 3, & \ldots \end{array}$$

$f: \mathbb{N} \to \mathbb{N}_0$

$f(n) = n - 1$

injective ✓

surjective ✓

$\therefore$ it is bijective!

for any $c \in \mathbb{N}_0$,

$f(c+1) = c$

Countably Infinite

$f : S \to \mathbb{N}$

We say set S is **countably infinite** if and only if there exists a bijection $f : \mathbb{N} \to S$. If such a bijection does not exist, we say S is **uncountably infinite**.

- It turns out that our bijection can even be from any other countably infinite set, not just $\mathbb{N}$.
- One way to think of this is to give each number a waiting number in an infinitely long line! We are essentially finding an **ordering** of S.

$f : \mathbb{N}_0 \to S$

$\mathbb{N}$	$\mathbb{N}_0$
1	0
2	1
3	2
$\vdots$	$\vdots$

positions in a line!

e.g. 0100101

Example: Bitstrings

A bitstring is a number written in binary, i.e. a sequence of 0s and 1s.

a) Consider the set of all *bitstrings* with length n. Is this set finite, countably infinite, or uncountably infinite?

b) Now consider the set of all bitstrings of finite length. Is this set finite, countably infinite, or uncountably infinite?

A C B

Attendance

tinyurl.com/wewantad