Announcements

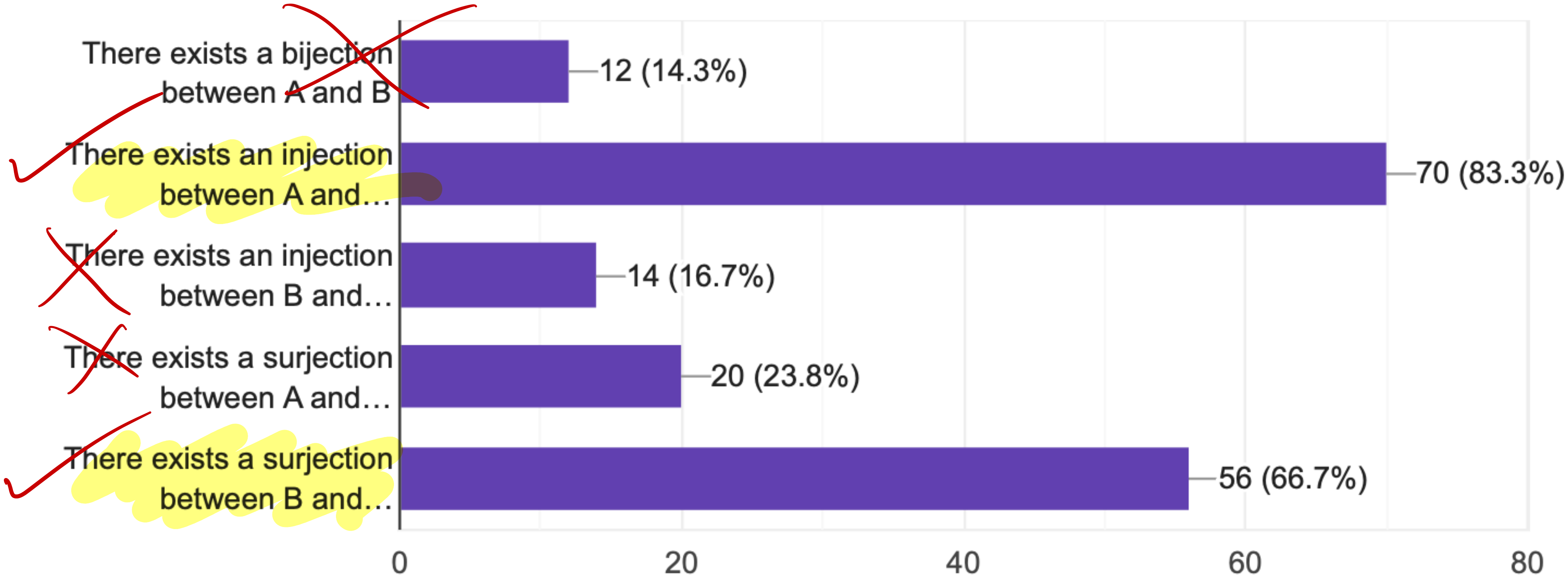
- Homework 1 due tomorrow; Gradescope code is on website if you're not already added
- Quiz 1 is a week from today!
- Would highly recommend looking at the textbook section for 1.3.

In-lecture quiz from Tuesday

$|A| < |B|$

Suppose A and B are finite sets, and suppose that A is a proper subset of B. Select all that apply:

84 responses



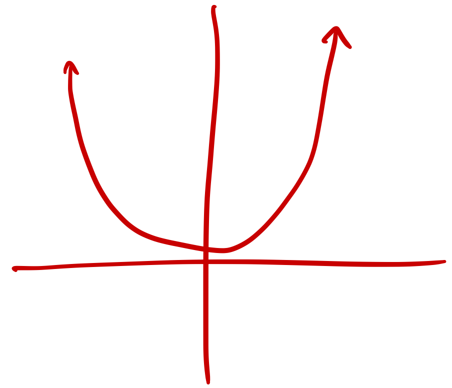
Clarifications

1. We say that two sets A, B the same cardinality if and only if **there exists** a bijection $A \to B$ between the two sets.
   - We aren't saying that every single function $f : A \to B$ is a bijection; instead, we're saying that we are **able to find** a bijection
2. To prove that a function is a bijection, we must prove that it's **both** an injection and a surjection!
   - To prove that a function is an injection, we need to show that no two inputs map to the same output.
   - To prove that a function is a surjection, we need to show that for any $c \in \text{Codomain}$, there is a specific $b \in \text{Domain}$ such that $f : b \mapsto c$.

$A = \{1, 2\}$
$B = \{3, 4\}$

$f(2) = 3$
$f(1) = 4$

$f(1) = f(2) = 4$



naturals are proper subset of wholes

Natural Numbers and Whole Numbers

The **natural numbers** (also known as the counting numbers), denoted by $\mathbb{N}$, are the most primitive numbers; ones that occur trivially in nature that can be used to count a (non-zero) number of things.

$$\mathbb{N} = \{1, 2, 3, 4, ...\}$$

Then, the set of **whole numbers**, denoted by $\mathbb{N}_0$, is the union of the set of counting numbers with the number 0.

↓ also c.i.

$$\mathbb{N}_0 = \{0, 1, 2, 3, 4, ...\} = \{0\} \cup \mathbb{N}$$

$\mathbb{N}$	$\mathbb{N}_0$
1 →	0
2 →	1
3	2
4	3
5	4
6	5
$\vdots$	$\vdots$

Last class, we showed the bijection $\mathbb{N} \to \mathbb{N}_0$ is given by

$$f : x \mapsto x - 1$$

Countably Infinite

We say set S is **countably infinite** if and only if there exists a bijection $f : \mathbb{N} \to S$. If such a bijection does not exist, we say S is **uncountably infinite**.

- It turns out that our bijection can even be from any other countably infinite set, not just $\mathbb{N}$.
- We can even find a bijection in the reverse direction, i.e. $g : S \to \mathbb{N}$, since all bijections are invertible
- One way to think of this is to give each number a waiting number in an infinitely long line! We are essentially finding an **ordering** of S.
- We showed that there exists a bijection from $\mathbb{N} \to \mathbb{N}_0$, telling us that $\mathbb{N}_0$ is also countably infinite
- There is a more general term that we use to refer to finite and countably infinite sets – we say these sets are **countable**
    - On the other hand, we say uncountably infinite sets are **uncountable**

e.g. 01101

Example: Bitstrings

A bitstring is a number written in binary, i.e. a sequence of 0s and 1s.

a) Consider the set of all *bitstrings* with length n. Is this set finite, countably infinite, or uncountably infinite? for one specific value of n

e.g. given $n=8$

$S = \{00000000, \ldots\ldots 11111110, 11111111\}$

$\boxed{\text{finite}}$

b) Now consider the set of all bitstrings of finite length. Is this set finite, countably infinite, or uncountably infinite?

for all values of $n \in \mathbb{N}$

$\mathbb{N}$	1	2	3	...
S	1	10	11	...

bijection

One-to-one correspondance

$\boxed{\text{countably infinite}}$

Integers

The set of integers, denoted by $\mathbb{Z}$, is the union of the whole numbers with their negatives

$$\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$

Are the integers countably infinite, or uncountably infinite?

$\mathbb{N}_0$	0	1	2	3	4	5	6	7	8	$\cdots$
$\mathbb{Z}$	0	1	-1	2	-2	3	-3	4	-4	$\cdots$

$g : \mathbb{Z} \to \mathbb{N}_0$

$$g(z) = \begin{cases} -2z & \text{if } z \leq 0 \\ 2z - 1 & \text{if } z > 0 \end{cases}$$

$f : \mathbb{N}_0 \to \mathbb{Z}$

$$f(n) = \begin{cases} -\dfrac{n}{2} & \text{if } n \text{ is even} \\ \dfrac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

this is a bijection!

$\therefore$ $\mathbb{Z}$ are countably infinite

Rational Numbers

The set of rational numbers, denoted by $\mathbb{Q}$, is the set of all possible combinations of one integer divided by another, with the latter integer being non-zero.

$$\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$$

Are the rational numbers countably infinite, or uncountably infinite?

Dense orderings

for all

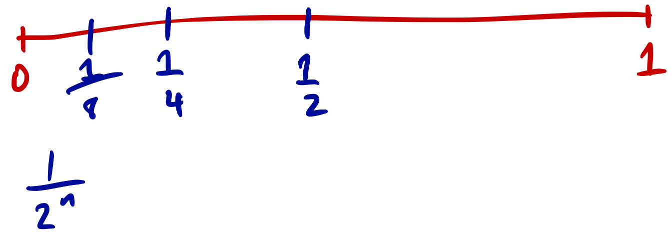
exists

such that

We say a set S is **densely-ordered** if it satisfies the following property:

$$\forall a, b \in S, \exists c \in S : a < c < b$$

In English, this condition states that between any two elements in a set, there is some other element in the set that is between the first two.



The set of rational numbers is densely ordered. Suppose m and n are two rational numbers, such that $m < n$. Then, $\frac{m+n}{2}$ is a rational number that is is between the two.

$a_1, b_1, a_2, b_2 \in \mathbb{Z}$

$m = \frac{a_1}{b_1}$ $\quad n = \frac{a_2}{b_2}$

$$\frac{m+n}{2} = \frac{\frac{a_1}{b_1} + \frac{a_2}{b_2}}{2} = \boxed{\frac{a_1 b_2 + a_2 b_1}{2 b_1 b_2}}$$

What implications does this have on the countability of $\mathbb{Q}$?

rationals ARE countably infinite!

$\mathbb{N} \subset \mathbb{Q}$

$|\mathbb{N}| = |\mathbb{Q}|$

$x \geq y$

$y \geq x$

$\Rightarrow x = y$

$\rightarrow |\mathbb{N}| \leq |\mathbb{Q}|$ injection $\mathbb{N} \to \mathbb{Q}$

$f(n) = \frac{n}{1}$

$\rightarrow |\mathbb{Q}| \leq |\mathbb{N}|$

injection $\mathbb{Q} \to \mathbb{N}$

$\mathbb{Q}$	$\mathbb{N}$
⋮	⋮

injection

$\mathbb{N} \subset \mathbb{Q}$

$\mathbb{N} = \{1, 2, 3, 4, \ldots\} \rightarrow \frac{n}{1}$ $\quad \frac{23}{1}$

$\mathbb{Q} = \{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \}$

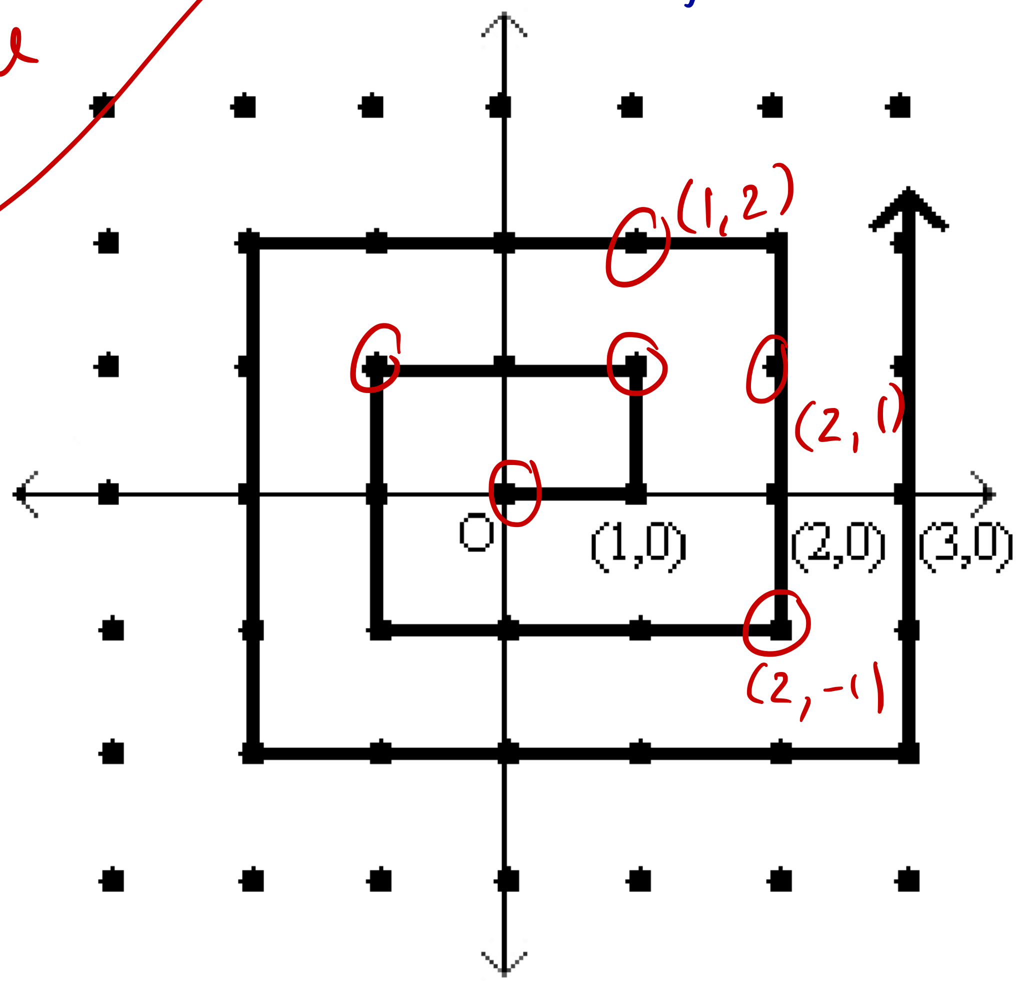
Injection from $\mathbb{Q} \to \mathbb{N}$:

perfectly valid function!

$(a, b) \mapsto \frac{b}{a}$

$\mathbb{Q}$	$\mathbb{N}$
0	1
1	2
-1	3
$-1/2$	4
$1/2$	5
2	6
	7
	8
	⋮

$\mathbb{N} \to$ positions in spiral



To recap:

- Since $\mathbb{N} \subseteq \mathbb{Q}$, we have $|\mathbb{N}| \leq |\mathbb{Q}|$. $f: n \mapsto \frac{n}{1}$
- Since there is an injection $\mathbb{Q} \to \mathbb{N}$, we have $|\mathbb{Q}| \leq |\mathbb{N}|$.
- Therefore, we have $|\mathbb{Q}| = |\mathbb{N}|$, meaning that the **rationals are countably infinite**.

Real Numbers

The set of real numbers, denoted by $\mathbb{R}$, is the set of all possible distances from 0 on a number line.

$$\mathbb{R} = \{3, \pi, -\sqrt{63}, 0.1224, \frac{2}{3}, ...\}$$

For the sake of completeness, we define the irrationals.

Irrational Numbers

The set of irrational numbers, denoted by $\mathbb{R} \backslash \mathbb{Q}$, is the set of real numbers that are not rational. That is, they are real numbers that cannot be written as an integer divided by another integer.

$$\mathbb{R} \backslash \mathbb{Q} = \{\pi, -e, \sqrt{5}, ...\}$$

Are the real numbers countable?

No.

The proof of this is beyond the scope of this course.

The argument used to prove that $\mathbb{R}$ is uncountably infinite is called Cantor's Diagonalization.

$x^2 - 2 = 0$

Transcendental Numbers

We say a number is **transcendental** if it is not the solution to a non-zero polynomial equation with integer coefficients.

In other words, t is transcendental if there are no $a_0, a_1, \ldots, a_n$ such that

$$a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0 = 0$$

Common examples of transcendental numbers are π and e. On the other hand, while $\sqrt{2}$ is irrational, it is not transcendental, as it is a root to $x^2 - 2 = 0$.

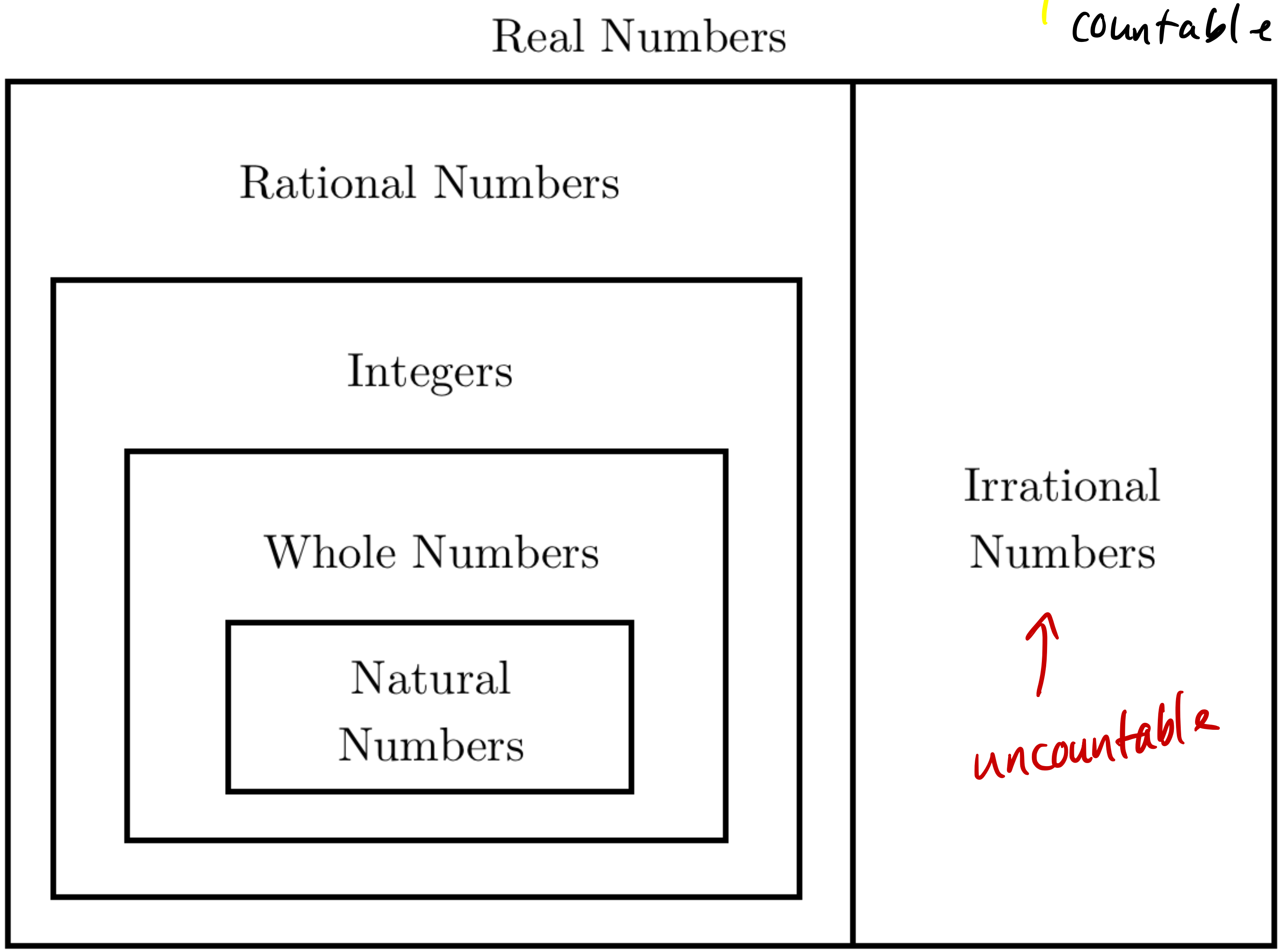
No rational number is transcendental; if $m = \frac{a}{b} \in \mathbb{Q}$, we have that m is a solution to $bx - a = 0$. As a consequence, all real transcendental numbers are irrational, but transcendental numbers can also be complex.

If a number is not transcendental, i.e. if it *is* the solution to a non-zero polynomial with integer coefficients, we call it **algebraic**.

algebraic $\cup$ (real) transcendental $=$ real

$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

countable



The complex numbers appendix of the textbook has a more complete picture. However, since the complex numbers are a superset of the reals, we know that complex numbers are uncountably infinite as well.

Key Takeaways

- Not all infinite sets have the same cardinality! There are "different levels" of infinite-ness
- For our purposes, we will simply classify infinite sets as either **countably infinite** or **uncountably infinite**
- The sets $\mathbb{N}$,$\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$ are all countably infinite. $\mathbb{R}$ is not countably infinite.
- If S is a set with infinitely many elements, there is no notion of $|S|$ – we can only look at $|S|$ relative to the cardinality of some other infinite set

$|\mathbb{N}| = |\mathbb{Q}| = |\mathbb{Z}|$

Attendance

tinyurl.com/freefultz

Propositional Logic

Set theory and propositional logic are intertwined. We need to formally study propositional logic before discussing proof techniques (which we will begin later next week).

Corresponds to 1.4 in our book. 1.5 gives a nice summary of all of the notation that we'll see in the following section.

Propositions

truth value

A proposition is a statement that is has a definitive value - either true or false.

Are the following statements propositions?

- "13 is prime" ✓
- "x is prime" : proposition $P(x)$ ✓
- "it is 93 degrees outside right now in Berkeley, CA" ✓
- "LeBron James is the greatest basketball player of all time" ✗

although I think it's true
... not a prop!

A, B propositions

$A \wedge B$

Logical Operators

Logical operators allow us to form complex propositions. These form the basis of everything we'll see in propositional logic.

1. **Conjunction**: $A \wedge B$, read "A and B"

   $A \cap B = \{x : x \in A \wedge x \in B\}$ intersection

   both must be true

2. **Disjunction**: $A \vee B$, read "A or B"

   $A \cup B = \{x : x \in A \vee x \in B\}$

   at least one must be true

3. **Negation**: $\neg A$, read "not A"

   $A^c = \{x : \neg(x \in A)\}$

A	B	$A \wedge B$	$A \vee B$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

truth table

We can use conjunctions, disjunctions and negations to create more complicated logical statements.

$$U(x) \wedge (E(x) \vee \neg P(x))$$

$U(x)$: x is ≤ 100

$E(x)$: x is even

$P(x)$: x is prime