# Lecture 5: Propositional Logic

Introduction to Mathematical Thinking

February 12th, 2019

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# Announcements

- **Quiz 1 on Thursday!**
    - 3:40-4:10 in class. Reach out if you require DSP accommodations.
    - Will cover content from Lecture 1 until today, roughly corresponding to Chapter 1 in our book.
    - **No cheatsheets.**
    - Highly recommend attempting Homework 2 before the quiz, and looking at last semester's midterm (linked on the website).
    - Homework 1 walkthrough video also exists.
- Add/drop deadline is tomorrow! Make sure to enroll in the units for the course.

## Propositions and Logical Operators

A proposition is a statement that is has a definitive value - either true or false. Logical operators allow us to form complex propositions. These form the basis of everything we'll see in propositional logic.

Our building blocks are the **conjunction**, **disjunction**, and **negation** operations.

**Conjunction**: $A \wedge B$, read "$A$ and $B$", is only when both $A$ and $B$ are true.

In set theory,

$$A \cap B = \{x : x \in A \wedge x \in B\}$$

| $A$ | $B$ | $A \wedge B$ |
| --- | --- | --- |
| True | True | True |
| True | False | False |
| False | True | False |
| False | False | False |

truth table

**Disjunction**: $A \vee B$, read "$A$ or $B$", is true when either $A$ or $B$ is true (i.e. when at least one is true).

In set theory,

$$A \cup B = \{x : x \in A \vee x \in B\}$$

| $A$ | $B$ | $A \vee B$ |
|---|---|---|
| True | True | True |
| True | False | True |
| False | True | True |
| False | False | False |

allows both to be true!
(unlike conversational English)

**Negation**: $\neg A$, read "not $A$", is true when $A$ is false.

In set theory,

$$A^c \ \cancel{\neg A} = \{x : \neg(x \in A)\}$$

$$\{x : x \notin A\}$$

| $A$ | $\neg A$ |
| --- | --- |
| T | F |
| F | T |

For example, suppose we have three propositions $U, P, E$ defined by $U(x) \equiv x \text{ is under } 30$, $P(x) \equiv x \text{ is prime}$, and $E(x) \equiv x \text{ is even}$. Then,

$$U(x) \wedge (\neg P(x) \vee \overset{E}{\cancel{Q}}(x))$$

is a proposition that reads "$x$ is under 30 and [redacted] not prime or even".

## De Morgan's Laws, Revisited

De Morgan's Laws allow us to simplify and re-write negations. Recall, we saw in Set Theory that the following rules apply for sets $A, B$:

$$(A \cap B)^C = A^C \cup B^C$$

$$(A \cup B)^C = A^C \cap B^C$$

It turns out that the same rules apply for propositions $P, Q$!

$$\neg(P \land Q) \equiv \neg P \lor \neg Q$$

$$\neg(P \lor Q) \equiv \neg P \land \neg Q$$

equivalent

We can prove De Morgan's laws using **truth tables**. We use truth tables to **prove** that two logical expressions are **equivalent**, i.e. that upon the same inputs, they produce the same outputs.

Proof of $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$:

| $P$ | $Q$ | $\neg(P \wedge Q)$ | $\neg P \vee \neg Q$ |
|---|---|---|---|
| T | T | F | F |
| T | F | T | T |
| F | T | T | T |
| F | F | T | T |

| $P \wedge Q$ |
|---|
| T |
| F |
| F |
| F |

truth table

$\therefore \neg(P \wedge Q) \equiv \neg P \vee \neg Q$

| $P$ | $Q$ | $\neg(P \vee Q)$ | $\neg P \wedge \neg Q$ |
|---|---|---|---|
| T | T | F | F |
| T | F | F | F |
| F | T | F | F |
| F | F | T | T |

$E$ : $t$ is even    $I$ : $t$ is irrational

$P$ : $t$ is prime

## Example

What is the negation of the statement, "$t$ is even and not prime or irrational and prime"?

*Hint: First, start by re-writing the statement in propositional logic.*

original: $(E \wedge \neg P) \vee (I \wedge P)$

negation
$$\neg\big((E \wedge \neg P) \vee (I \wedge P)\big)$$
$$\equiv \neg(E \wedge \neg P) \wedge \neg(I \wedge P)$$
$$\equiv (\neg E \vee \neg\neg P) \wedge (\neg I \vee \neg P)$$
$$\equiv (\neg E \vee P) \wedge (\neg I \vee \neg P)$$

$\neg(A \vee B)$
$\equiv \neg A \wedge \neg B$

$\neg(A \wedge B)$
$\equiv \neg A \vee \neg B$

$t$ is not even or prime, and not irration or not prime

# Exclusive OR

Consider our regular "OR" operation:

| $A$ | $B$ | $A \vee B$ |
|---|---|---|
| True | True | True |
| True | False | True |
| False | True | True |
| False | False | False |

$A \oplus B$, read "$A$ xor $B$", is true when exactly one of $A, B$ is true.

| $A$ | $B$ | $A \oplus B$ |
|---|---|---|
| True | True | False |
| True | False | True |
| False | True | True |
| False | False | False |

Rewrite $A \oplus B$ in terms of

$\vee, \wedge, \neg$ ?

$\sim$

$(A \vee B) \wedge \neg (A \wedge B)$

$(A \vee B) \wedge \underbrace{(\neg A \vee \neg B)}_{\text{equivalent, using De Morgan's}}$

Claim: $A \oplus B \equiv (A \vee B) \wedge \neg(A \wedge B)$.

Proof:

| $A$ | $B$ | $A \oplus B$ | $(A \vee B) \wedge \neg(A \wedge B)$ |
| --- | --- | --- | --- |
| True | True | False | False |
| True | False | True | True |
| False | True | True | True |
| False | False | False | False |

if P, then Q

## Implications

$P \Rightarrow Q$, read "$P$ implies $Q$", says that if $P$ is true, then $Q$ must be true; if $P$ is false, it says nothing about $Q$ ($Q$ could either be true or false).

For example, **if** all we know about today's date is that it's Christmas, **then** we also know that the current month is December. However, if we don't know that it's Christmas, then it may or may not be December. We could then say:

$$\underbrace{\text{today is Christmas}}_{P} \Rightarrow \underbrace{\text{the current month is December}}_{Q}$$

| $P$ | $Q$ | $P \Rightarrow Q$ |
|---|---|---|
| True | True | True |
| True | False | False |
| False | True | True |
| False | False | True |

truth table defines implication

↑ just a truth value!

implications are just expressions with a truth value

Another interpretation of the implication:

Suppose I make the promise to you that **if it rains tomorrow, I will give you 100 dollars.** The "truth value" of this implication is whether or not I held my promise:

- If it rains tomorrow, and I give you 100 dollars, I held my promise! (T)
- If it rains tomorrow, and I don't give you 100 dollars, I did not hold my promise. (F)
- If it does not rain tomorrow, and I give you 100 dollars, I held my promise! (T)
- If it does not rain tomorrow, and I don't give you 100 dollars, I still held my promise! (T)

$\neg P \vee Q$

**Can we rewrite $P \Rightarrow Q$ in terms of $\vee, \wedge, \neg$?**

$(P \wedge Q) \vee \neg P$

$$\underbrace{x \text{ is even}}^{P} \implies \overbrace{x^2 \text{ is even}}^{Q}$$

$$\equiv \underbrace{x \text{ is odd}}_{\neg P} \; \underbrace{\text{or}}_{\vee} \; \underbrace{x^2 \text{ is even}}_{Q}$$

We claim that $P \Rightarrow Q \equiv \neg P \vee Q$.

| $P$ | $Q$ | $P \to Q$ | $\neg P \vee Q$ |
|---|---|---|---|
| True | True | True | True |
| True | False | False | False |
| False | True | True | True |
| False | False | True | True |

Since $P \Rightarrow Q$ and $\neg P \vee Q$ produce the same outputs on all combinations of inputs, we can say the two expressions are equivalent.

# Example

$P \implies Q$

Determine the negation of the following statement:

"if $\underbrace{t^2 \text{ is prime}}_{P}$, then $\underbrace{t \text{ is not an integer}}_{Q}$"

$$P \implies Q \equiv \neg P \vee Q$$

$$\neg(P \implies Q) \equiv \neg(\neg P \vee Q)$$

$$\equiv \neg\neg P \wedge \neg Q \equiv \boxed{P \wedge \neg Q}$$

$t^2$ is prime and $t$ is an integer

$P \qquad \wedge \qquad \neg Q$

## Negation of an Implication

In general, the negation of an implication of the form $P \Rightarrow Q$ is $P \wedge \neg Q$. We can derive this using De Morgan's Laws:

$$\neg(P \Rightarrow Q) \equiv \neg(\neg P \vee Q)$$

$$\equiv (\neg\neg P) \wedge (\neg Q)$$

$$\equiv \boxed{P \wedge \neg Q}$$

**This is an extremely important result.** We wil use it extensively when we talk about proofs.

## Extensions of the Implication

We'll now look at two common logical expressions that are related to the standard implication, $P \Rightarrow Q$:

- The **contrapositive**, $\neg Q \Rightarrow \neg P$
- The **converse**, $Q \Rightarrow P$

## Contrapositive

$P \Rightarrow Q$

$A \Rightarrow B \equiv \neg A \vee B$

The contrapositive, $\neg Q \Rightarrow \neg P$, of an implication is actually logically equivalent to the implication itself!

| $P$ | $Q$ | $P \Rightarrow Q$ | $\neg Q \Rightarrow \neg P$ |
|---|---|---|---|

$$\neg Q \Rightarrow \neg P \equiv \neg(\neg Q) \vee (\neg P) \equiv Q \vee \neg P$$

$$\equiv \neg P \vee Q$$

$$\equiv P \Rightarrow Q$$

If a statement reads "if $P$, then $Q$", its contrapositive reads "if not $Q$, then not $P$."

Examples:

| Statement | Contrapositive |
|---|---|
| If it is sunny outside, I will wear my sunglasses. | If I don't wear my sunglasses, it is not sunny outside. |
| If it is a square, it is a rectangle. | If it is not a rectangle, it is not a square. |

A statement and its contrapositive mean the exact same thing. *Why do we need it, then?*

## Converse

$P \Rightarrow Q$

The converse, $Q \Rightarrow P$ of an implication, unlike the contrapositive, is **not equivalent** to the original implication.

If a statement reads "if $P$, then $Q$", its converse reads "if $Q$, then $P$."

Examples:

| Statement | Converse |
|---|---|
| If it is sunny outside, I will wear my sunglasses. | If I wear my sunglasses, it is sunny outside. |
| If it is a square, it is a rectangle. | If it is a rectangle, it is a square. |
| If $f$ is a bijection, it is both an injection and surjection. both are true! | If $f$ is both an injection and surjection, it is a bijection. |
| If today is Christmas, it is currently December. | If it is currently December, then today is Christmas. |

Examples 1, 2, and 4 are examples of a converse not being equivalent to the original statement. However, in example 3, we have that the two statements mean the same thing; this is a special case.

## Example

$P \Rightarrow Q$

converse $Q \Rightarrow P$

contrapositive $\neg Q \Rightarrow \neg P$

Determine the contrapositive and converse of the statement,

"if $\underbrace{p \text{ and } q \text{ are prime}}_{P}$, then $\underbrace{(\gcd(p, q) = 1 \text{ or } p = q.)}_{Q}$

*Hint: You will need to use De Morgan's Laws to negate the second half of the implication when finding the contrapositive.*

contrapositive: if $\gcd(p,q) \neq 1$ and $p \neq q$, then $p$ is not prime or $q$ is not prime

converse: if $\gcd(p,q)=1$ or $p=q$, then $p$ and $q$ are prime.

Injection

$\underbrace{f(x_1)=f(x_2)}_{P} \Rightarrow \underbrace{x_1 = x_2}_{Q}$

$\underbrace{x_1 \neq x_2}_{\neg Q} \Rightarrow \underbrace{f(x_1) \neq f(x_2)}_{\neg P}$

a function is bijective iff it is both surjective and injective

## If and only if

$A \Leftrightarrow B$, read "$A$ if and only if $B$", says that $A$ is true only when $B$ is true, and $A$ is false only when $B$ is false - in other words, that $A$ **and** $B$ **are equivalent**.

Another interpretation of this is, $A$ iff $B$ is true when both $A \Rightarrow B$ and $B \Rightarrow A$, i.e. when both an implication and its converse are true.

**For example**, "it is Christmas if and only if it is December 25th" decomposes into:

1. "it is Christmas if it is December 25th" if it is Dec 25, then it is Christmas
2. "it is December 25th if it is Christmas" if it is Christmas, then it is Dec 25

This is just a fancy way of saying "Christmas is on December 25th".

$A \equiv B$

Consider the following truth table:

$A \Leftrightarrow B \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$

| $A$ | $B$ | $A \Leftrightarrow B$ | $(A \Rightarrow B) \wedge (B \Rightarrow A)$ |
|---|---|---|---|
| True | True | True | True |
| True | False | False | False |
| False | True | False | False |
| False | False | True | True |

e.g., Today being Christmas implies that today is December 25th, and today being December 25th implies that today is Christmas - since these two propositions imply one another, we can say they're equivalent.

**Important for proof techniques!** When we end up proving a statement of the form "if and only if", we will have to prove both directions, i.e. $A \Rightarrow B$ and $B \Rightarrow A$.

# Existential and Universal Quantifiers

- **universal quantifier** $\forall$, read "for all"
- **existential quantifier** $\exists$, read "there exists"

$\forall x \; P(x)$ : for all $x$, $P(x)$ is true

Recall: $f : A \to B$ is surjective ~~when~~ iff $\forall b \in B, \exists a \in A : b = f(a)$.

e.g. Suppose $E(x)$: "$x$ is even" and $U(x)$: "$x$ is odd." What do the following statements mean?

$$\forall x \in \mathbb{N}, E(x) \vee U(x)$$

$\to$ for all naturals $x$, $x$ is even or $x$ is odd (true)

$$\forall x \in \mathbb{N}, E(x) \wedge U(x)$$

for all naturals $x$, $x$ is even and $x$ is odd (false)

# De Morgan's Laws for Quantifiers

De Morgan's Laws allow us to change a universal quantifier into an existential quantifier, and vice versa.

not all students are sophomores

exists a student that is not a sophomore

$$\neg(\forall x\, P(x)) \equiv \exists x\, \neg P(x)$$

$$\neg(\exists x\, P(x)) \equiv \forall x\, \neg P(x)$$

all students are not sophomores / no student is a sophomore

does not exist a sophomore

For example, suppose $P(x)$: "$x$ is prime."

1. "~~If i~~t is not true that $x$ is prime for all natural numbers $x$, ~~then~~ there must exist some natural number $x$ that is not prime."
2. "~~If~~ it is not true that there exists a natural number $x$ that is prime, ~~then~~ for all natural numbers $x$, $x$ is not prime."

Now that we've seen all forms of De Morgan's Laws, we can find the negations of complicated statements. For example, the statement

$$\forall q \in \mathbb{Q}, (\exists t \in Q : qt = 1 \vee q = 0)$$

reads "all rational ~~natural~~ numbers $q$ have an inverse or are equal to 0." Its negation,

taking negation

$\forall \longleftrightarrow \exists$

$\vee \longleftrightarrow \wedge$

$= \longleftrightarrow \neq$

$$\exists q \in Q, (\forall t \in Q : qt \neq 1 \wedge q \neq 0)$$

reads "there exists some non-zero rational number $q$ that does not have an inverse."

We can propagate a negation through a long chain of propositions and quantifiers:

$$\neg(\forall x \, \exists y : P(x,y) \vee Q(y))$$

$$\equiv \exists x \forall y : \neg P(x,y) \wedge \neg Q(y)$$

**Note:** You cannot arbitrarily reverse the order of $\forall$ and $\exists$.

For example, suppose $P(x, y)$ is the proposition $x > y$. Then,

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} : P(x, y)$$

states that for all integers $x$, there exists an integer $y$ that is smaller than $x$. However,

$$\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z} : P(x, y)$$

states that there is some integer $y$ that is smaller than every other integer $x$. This is saying something very different (there is no smallest integer, whereas the first statement is true).

## Sample Problem from Last Semester's Midterm

We want to be able to create our own logical operators and show that they are **complete**, i.e. that we can recreate $\vee$, $\wedge$ and $\neg$ using our new operation.

Consider the NOR operation, $A \downarrow B$, which is true only when $A, B$ are both false:

| $A$ | $B$ | $A \downarrow B$ |
|---|---|---|
| T | T | F |
| T | F | F |
| F | T | F |
| F | F | T |

$A \vee B$

T

T

T

F

$A \downarrow B$ in terms of

$\vee, \wedge, \neg$ ?

Part a: Can you write $A \downarrow B$ in terms of $\vee, \wedge, \neg$?

$$A \downarrow B \equiv \neg (A \vee B)$$

Prove : truth table

Part b: Can you write $\neg A$ in terms of $\downarrow$?

$A \downarrow A$

Part c: Can you write $A \vee B$ in terms of $\downarrow$?

$$A \downarrow B \equiv \neg (A \vee B)$$

$$\neg (A \downarrow B) \equiv \neg\neg (A \vee B) \equiv A \vee B$$

$$A \vee B \equiv (A \downarrow B) \downarrow (A \downarrow B)$$

Bonus: Can you write $A \Rightarrow B$ in terms of $\downarrow$?