Lecture 7.5: Stars and Bars

http://book.imt-decal.org, Ch. 4 (in progress)

Introduction to Mathematical Thinking

October 29th, 2018

Suraj Rampure

Announcements

- Homework 7 due Wednesday
- Last Wednesday's video didn't record audio for some reason... sorry, will make sure this doesn't happen again

Stars and Bars

How many ways can I distribute 12 pieces of candy to 3 of my friends?

- 12 "stars", or items
- 3 - 1 = 2 "bars", or separators

For example:

$$\underbrace{\ast \ast} \mid \underbrace{\ast \ast \ast \ast \ast \ast} \mid \underbrace{\ast \ast \ast \ast}$$

This setup represents friend 1 getting 2 pieces, friend 2 getting 6 pieces and friend 3 getting 4 pieces.

$$\underbrace{\ast} \mid \mid \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast \ast$$

This setup represents friend 1 getting 1 piece, friend 2 getting 0 pieces and friend 3 getting 11.

This problem boils down to finding the number of permutations of

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$$\frac{11!}{4!4!2!}$$

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In our case, this is $\frac{14!}{12!2!} = \binom{14}{2} = \binom{14}{12} = \binom{14}{2}$

In general, the number of ways to arrange n **indistinguishable (i.e. identical) items** into k **distinguishable bins** is

$$\binom{a+b}{a} = \frac{(a+b)!}{a!\,b!} = \binom{a+b}{b}$$

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

1 : 6
2 : 5
3 : 1

Since k bins corresponds to $k-1$ bars, we can also write this number as

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{\text{stars} + \text{bars}}{\text{bars}} = \binom{\text{stars} + \text{bars}}{\text{stars}}$$

1 : 5
2 : 6
3 : 1

Example: Determine the number of non-negative integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 25$$

bars

This fits the same model. We have 25 "stars", and 4 "bars", meaning there exist

$$\binom{29}{4} \qquad \binom{25+4}{4}$$

solutions to this equation.

Followup: How many *positive* integer solutions exist to this equation?

Previously, we had that $x_1, x_2, x_3, x_4, x_5 \geq 0$. We now want $x_i > 0$, or equivalently, $x_i \geq 1$.

To account for this, we can define $x_i' = x_i - 1$. Now, the only constraint we have is $x_i' \geq 0$, which we already know how to solve (from the last slide).

$x_1 \geq 1$

$x_1 - 1 \geq 0$

$x_1' \geq 0$

The number of positive integer solutions to

$$x_1 + x_2 + x_3 + x_4 + x_5 = 25$$

$$(x_1' - 1) + (x_2' - 1) + \cdots + (x_5' - 1) = 25 - 5$$

is the same as the number of non-negative integer solutions to

$$x_1' + x_2' + x_3' + x_4' + x_5' = 25 - 5 = 20$$

which is $\binom{24}{4}$.

In general, the number of positive integer solutions to $\sum_{i=1}^{k} x_i = n$ is $\binom{n-1}{k-1}$.

$$x_1 + x_2 + \cdots + x_k = n \rightarrow \binom{n-1}{k-1}$$

How many ways can I distribute 12 pieces of candy to 3 of my friends, such that they all get at least one piece?

This is the same problem as finding the number of positive integer solutions to $x_1 + x_2 + x_3 = 12$ which from the previous slide we have as

$$\binom{12-1}{3-1} = \binom{11}{2} = 55$$

Attendance