# Lecture 8: Mathematical Induction

http://book.imt-decal.org, Ch. 2.2

## Introduction to Mathematical Thinking

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# Announcements

- Homework 3 due tomorrow, 11:59PM
- We'll heavily rely on sigma notation, $\sum_{i=1}^{n}$, when talking about induction. If you're shaky with sigma notation, look at the Appendix section on Sigma and Pi Notation in our book (http://book.imt-decal.org)

- Quiz 2 is next Thursday!

**Example (for fun!)**

Prove $0.999... = 1$.

1

$$\begin{aligned} 10x &= 9.9999\ldots \\ - \quad x &= 0.9999\ldots \\ \hline 9x &= 9 \\ \boxed{x = 1} \end{aligned}$$

sum of an infinite geometric series $= \dfrac{\text{first term}}{1 - \text{ratio}} = \dfrac{\frac{1}{10}}{1-\frac{1}{10}} = \dfrac{\frac{1}{10}}{\frac{9}{10}} = \frac{1}{9}$

iff $|\text{ratio}| < 1$

2

$342 = 3 \cdot 10^2 + 4 \cdot 10^1 + 2 \cdot 10^0$

$0.342 = 3 \cdot 10^{-1} + 4 \cdot 10^{-2} + 2 \cdot 10^{-3}$

$= \frac{3}{10} + \frac{4}{100} + \frac{2}{1000}$

$\rightarrow$ $0.9999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$

$= \sum_{i=1}^{\infty} \frac{9}{10^i}$

$= 9 \sum_{i=1}^{\infty} \frac{1}{10^i}$

$= 9\left(\frac{1}{10} + \frac{1}{10} \cdot \frac{1}{10} + \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10} + \cdots\right)$

$= 9 \cdot \frac{1}{9} = \boxed{1}$

## Recap: Types of Proofs

- Direct Proofs
- Proof by Contradiction
- Proof by Contraposition
- Proof by Cases
- **Proof by Induction**

## Motivation

Suppose you're sitting in a massive lecture hall, and want to find out how many rows you're sitting from the front of the room. You *could* sit there and count, but consider this basic principle:

- The person sitting in the first row knows their row number *by default*: they're in the first row!
- If one knows the row number of the person in front of them, they add 1 to get their own row number

## Mathematical Induction

We use induction to prove properties about **all natural numbers**. Induction has three steps:

**1. Base Case**: Establish that the statement holds for $n = 0$ or $n = 1$ (or whatever makes the most sense in the situation) usually very simple

**2. Induction Hypothesis**: Assume that the statement holds true for $n = k$, for some arbitrary $k$

**3. Induction Step**: Given the fact that the statement holds true for $n = k$, show that it holds for $n = k + 1$ where most of the work will be

$P(k) \Rightarrow P(k+1)$

$P$ : proposition

*What does this remind you of from CS 61A? What are the parallels?*

recursion !

sum of first n natural numbers

## Example

Prove that $\sum_{i=1}^{n} i = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$.

① Base Case

$n=1$

| LS | RS |
|---|---|
| $\sum_{i=1}^{1} i = \boxed{1}$ | $\frac{1(2)}{2} = \boxed{1}$ |

LHS=RHS, ∴ base case holds

② Induction Hypothesis

Assume $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$,

for some arbitrary $k \in \mathbb{N}$.

③ Induction step

want to show $\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$

start with $\sum_{i=1}^{k+1} i$

$$\sum_{i=1}^{k+1} i = \underbrace{1 + 2 + \cdots + k} + k + 1$$

$$= \sum_{i=1}^{k} i + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1) \cdot \frac{2}{2}$$ by induction hypothesis

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

$F \Rightarrow \bigcirc$ always true

$\rightarrow$ we need to prove base case!

## In terms of propositional logic:

in last example: $P(1)$

**Base Case**: Show $P(0)$ (or other base case) holds true

**Induction Hypothesis**: Assume $P(k)$ holds true for some arbitrary $k \in \mathbb{N}_0$

**Induction Step**: Show $P(k) \Rightarrow P(k+1), \forall k$.

$$\forall P \Big( P(0) \wedge \forall k \big( P(k) \Rightarrow P(k+1) \big) \Rightarrow \forall n \in \mathbb{N}_0 \big( P(n) \big) \Big)$$

where $P(\cdot)$ represents the proposition we are trying to prove.

showed that $P(k) \Rightarrow P(k+1)$

More explicitly:

$$P(0) \Rightarrow P(1) \Rightarrow P(2) \Rightarrow \ldots$$

showed the base to be true

## Example

$(re^{i\theta})^n = r^n e^{in\theta}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$i: i^2 = -1$

$R^{k+1} = R^k \cdot R$

De Moivre's Theorem states the following:

$$[R(\cos t + i \sin t)]^n = R^n(\cos nt + i \sin nt)$$

Prove De Moivre's Theorem (for $n \in \mathbb{N}_0$) using induction.

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

Base Case

$n=0$

| LS | RS |
|---|---|
| $(R(\cos t + i \sin t))^0$ | $R^0(\cos 0t + i \sin 0t)$ |
| $= 1$ | $= 1(1+0)$ |
| | $= 1$ |

LHS = RHS,

$\therefore$ base case holds.

Induction Hypothesis

Assume proposition holds for $n=k$, i.e.

assume

$[R(\cos t + i \sin t)]^k = R^k(\cos kt + i \sin kt)$

## Induction Hypothesis

Assume proposition holds for $n=k$, i.e.

assume

$$[R(\cos t + i\sin t)]^k = R^k(\cos kt + i\sin kt)$$

$$\square^{k+1} = \square^k \square^1$$

$i : i^2 = -1$

$$\cos((k+1)t) = \cos(kt+t) = \cos kt\cos t - \sin kt\sin t$$

$$\sin((k+1)t) = \sin(kt+t) = \sin kt\cos t + \cos kt\sin t$$

## Induction Step

$$[R(\cos t + i\sin t)]^{k+1} = [R(\cos t + i\sin t)]^k\,[R(\cos t + i\sin t)]$$

by ind. hypothesis

$$= R^k(\cos kt + i\sin kt)\,R(\cos t + i\sin t)$$

$$= R^{k+1}\left(\cos kt\cos t + i(\cos kt\sin t + \sin kt\cos t) + i^2\sin kt\sin t\right)$$

$$= R^{k+1}\left(\underline{\cos kt\cos t - \sin kt\sin t} + i\,\underline{(\sin kt\cos t + \cos kt\sin t)}\right)$$

$$= R^{k+1}\left(\cos(k+1)t + i\sin(k+1)t\right)$$

$$\boxed{\therefore \text{ induction holds!}}$$

## Example

The Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ is defined by the **recurrence relation** :

$$F_1 = 1, F_2 = 1$$

$$F_n = F_{n-2} + F_{n-1} \quad \forall n \geq 3, n \in \mathbb{N}$$

Prove that $\sum_{i=1}^{n} F_i = F_{n+2} - 1$.

**Base Case**

$n=1$

| LS | RS |
|---|---|
| $\sum_{i=1}^{1} F_i = F_1$ | $F_{1+2} - 1$ |
| $= 1$ | $= F_3 - 1$ |
| | $= 2 - 1$ |
| | $= 1$ |

$\therefore$ base case holds

**Induction Hypothesis**

Assume $\sum_{i=1}^{k} F_i = F_{k+2} - 1$ where $k \in \mathbb{N}$.

**Induction Step**

$$\sum_{i=1}^{k+1} F_i = \sum_{i=1}^{k} F_i + F_{k+1}$$

by I.H. $= F_{k+2} - 1 + F_{k+1}$

$$= \underbrace{F_{k+2} + F_{k+1}}_{} - 1$$

$$= \boxed{F_{k+3} - 1}$$

$\therefore$ induction holds.

induct on n

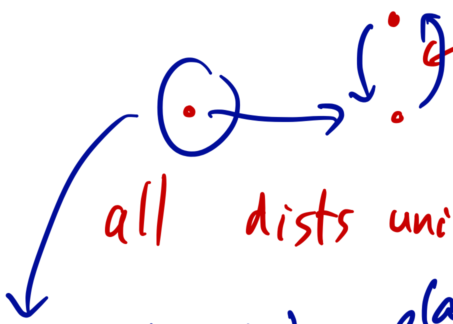
## Example

Suppose that there are $2n+1$ airports where $n$ is a positive integer. The distances between any two airports are all different. For each airport, there is exactly one airplane departing from it, and heading towards the closest airport. Prove by induction that there is an airport which none of the airplanes are heading towards.

**Base Case**

$n=1$

$\rightarrow 2(1)+1 = 3$ airports



shortest side

all dists unique

no incoming planes!

$\therefore$ the base case holds

**Induction Hypothesis**

Assume statement holds for $n=k$, i.e.

assume in $2k+1$ airports, there is an airport w/ no inc. planes

$\therefore$ induction holds!

**Induction Step**

Consider $2(k+1)+1$
$= 2k+3$ airports

→ consider the 2 airports closest to one another

→ they send planes to each other

→ consider remaining $2k+1$ arpts

→ by IH, one is empty

$9^n - 1 = 8c, \quad c \in \mathbb{Z}$

$8 | 24$

**Example**

Prove that $8 | 9^n - 1$, for $n \in \mathbb{N}$.

Base Case

$n = 1$

$9^1 - 1 = 8, \quad 8 | 8,$

$\therefore$ base case holds!

Induction Hypothesis

Assume $8 | 9^k - 1$, i.e.

$9^k - 1 = 8c, \quad c \in \mathbb{Z}$

$9^k = 8c + 1$

Induction step

Show $8 | 9^{k+1} - 1$, i.e. $9^{k+1} - 1 = 8b,$
$b \in \mathbb{Z}$

$$\begin{aligned} 9^{k+1} - 1 &= 9^k \cdot 9 - 1 \\ &= (8c+1) \cdot 9 - 1 \\ &= 9 \cdot 8c + 9 - 1 \\ &= 8 \cdot 9c + 8 = 8(9c+1) \end{aligned}$$

$\therefore$ induction holds!

## Next time, we will...

- Introduce the idea of "strong induction", where we assume more than just that $P(k)$ holds
- Analyze various inductive proofs and point out the flaws in them

Our textbook has several examples of induction problems, many of which we didn't cover in class, but that may be helpful for your next homework. Try and attempt each of these problems on your own before looking at the solution.