Lecture 9: Induction, Series and Sequences

http://book.imt-decal.org, Ch. 2.2, 2.3

Introduction to Mathematical Thinking

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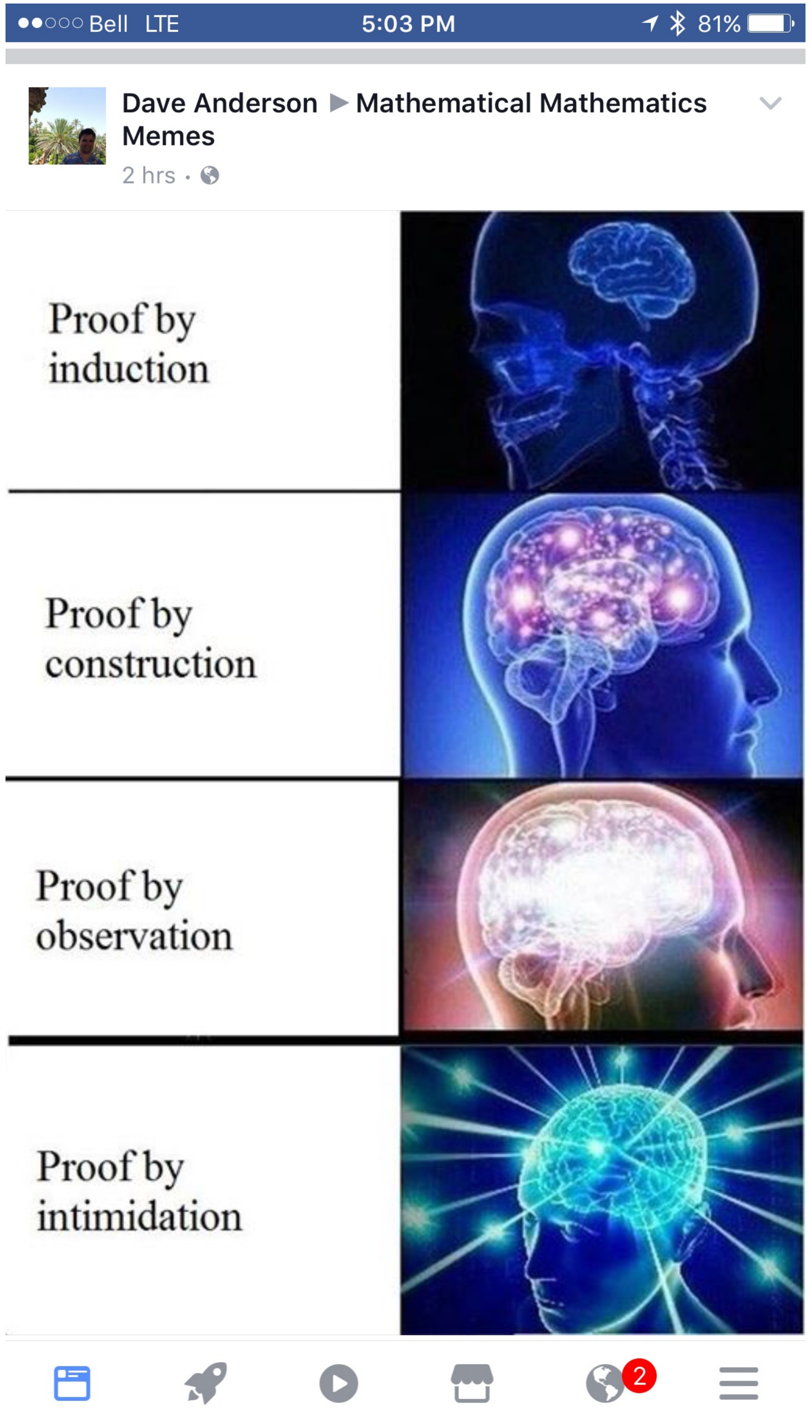
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Dave Anderson ▶ Mathematical Mathematics Memes
2 hrs ·
Proof by induction
Proof by construction
Proof by observation
Proof by intimidation
2

Announcements

- Quiz on Thursday, from 3:40-4:10PM
  - Will cover material since the last quiz, but technically everything is in scope
  - I know the course is moving quickly – the quizzes are more to keep you in check and to make sure you're doing the homework.
  - Feel free to reach out with conceptual questions, even if it's after the quiz

Recap: Proof by Induction

We use induction to prove properties about **all natural numbers**. Induction has three steps:

1. **Base Case**: Establish that the statement holds for $n = 0$ or $n = 1$ (or whatever makes the most sense in the situation)

2. **Induction Hypothesis**: Assume that the statement holds true for $n = k$, for some arbitrary k

3. **Induction Step**: Given the fact that the statement holds true for $n = k$, show that it holds for $n = k + 1$

$P(k) \implies P(k+1)$

Example

Consider the sequence defined by

$$t_0 = 1$$

$$t_n = 2t_{n-1} + 7, \forall n \in \mathbb{N}$$

Using induction on n, prove that $t_n \leq 2^{n+3} - 7$.

Base Case: $n = 0$

LS	RS
$t_0 = 1$	$2^{0+3} - 7$
	$= 8 - 7$
	$= 1$

$1 \leq 1$, $\therefore$ BC holds.

IH:
Assume $t_k \leq 2^{k+3} - 7$, for some $k \in \mathbb{N}$

IS:

$t_{k+1} = 2t_k + 7$

by IH $\leq 2(2^{k+3} - 7) + 7$

$= 2^{k+4} - 14 + 7$

$= 2^{k+4} - 7$

$\Rightarrow t_{k+1} \leq 2^{k+4} - 7$

$\therefore$, induction holds.

Example

Recall, the Fibonacci sequence is defined as $f_1 = 1, f_2 = 1, f_n = f_{n-2} + f_{n-1}$, for $n \geq 3$.

Suppose $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Prove that, for all $n \in \mathbb{N}, n \geq 2$,

$$A^n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$$

Base case : $n = 2$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$f_3 = 1 + 1 = 2$

$f_2 = 1$

$f_1 = 1$

$$= \begin{bmatrix} f_3 & f_2 \\ f_2 & f_1 \end{bmatrix}$$

$\therefore$ base case holds.

IH

Assume $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k = \begin{bmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{bmatrix}$

IS

apply IH

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} f_{k+2} & f_{k+1} \\ f_{k+1} & f_k \end{bmatrix}$$

$\therefore$ induction holds

Strong Induction

may have to prove multiple base cases

So far, in our induction hypothesis, we've assumed that $P(k)$ holds true, and in our induction step, we've been proving the implication $P(k) \Rightarrow P(k+1)$. However, we may need to assume more than just $P(k)$ in our induction hypothesis in order to complete a proof.

and

In **strong induction**, instead of assuming just $P(k)$, we assume $P(1) \wedge P(2) \wedge \ldots \wedge P(k)$, and use this to show that $P(k+1)$ holds. It turns out that this proof technique is identical to standard induction, but can be more powerful, as it allows us to assume more.

weak induction (or just induction)

$$P(k) \Longrightarrow P(k+1)$$

strong induction

$$P(1) \wedge P(2) \wedge \ldots \wedge P(k) \Longrightarrow P(k+1)$$

Example – Postage Stamp Problem

Suppose you have a collection of 4-cent and 5-cent postage stamps. Prove that, for any $n \in \mathbb{N}, n \geq 12$, you can make postage for exactly n cents.

4 cent stamps

5 cent stamps

In other words, prove

$$\forall n \in \mathbb{N}, n \geq 12, \exists\, a, b \in \mathbb{N}_0 \colon 4a + 5b = n$$

$4a+5b=n, \quad n \geq 12$

by strong induction:
can say
$k-3 = 4\square + 5\triangle$
$k+1 = 4(\square+1)+5\triangle$

Base Case

$n=12$: $4\cdot 3+5\cdot 0=12$

$n=13$: $4\cdot 2+5\cdot 1=13$

$n=14$: $4\cdot 1+5\cdot 2=14$

$n=15$: $4\cdot 0+5\cdot 3=15$

IH

Assume that we can make postage for all values $\underbrace{12, 13,}_{\text{strong induction}} \ldots k$.

IS

consider $k+1 \geq 16$

we know we can make postage for $n=(k+1)-4$ (by IH)

$k-3 = k+1-4 = 4a+5b$

$k+1 = 4(a+1)+5b$

$\therefore$ if we can make postage for $n=k-3$, we can make postage for $n=k+1$.

Weak Induction vs. Strong Induction

	Induction	Strong Induction
Base Case	Prove $P(1)$, or other necessary base case(s)	Prove $P(1)$, or other necessary base case(s)
Induction Hypothesis	Assume $P(k)$, for $k \in \mathbb{N}$	Assume $P(1) \land P(2) \land \ldots \land P(k)$, for $k \in \mathbb{N}$
Induction Step	Prove $P(k) \Rightarrow P(k+1)$	Prove $P(1) \land P(2) \land \ldots \land P(k) \Rightarrow P(k+1)$

Think about both forms of induction in terms of "knocking down dominos".

- Weak induction: If domino k falls, then domino $k+1$ falls
- Strong induction: If dominos 1 through k fall, then domino $k+1$ falls

Example

$-4+2=-2$

$4-2=2$

$3\cdot 2^{\square}+3\cdot 2^{\square}=3\cdot 2^{\square+1}$

$4(-1)^{k-1}+2(-1)^{k}=2(-1)^{k+1}$

$P(k+1)$

need $P(k)$, $P(k-1)$

Consider the sequence defined by

$$a_1 = 1, a_2 = 8$$

$$a_n = a_{n-1} + 2a_{n-2}, n \geq 3$$

Prove that $a_n = 3 \cdot 2^{n-1} + 2(-1)^n, \forall n \in \mathbb{N}$. *Hint: Think about the number of base cases you need.*

Base Case

$n=1$

$a_1 = 3\cdot 2^0 + 2(-1)^1 = 3-2 = 1$ ✓

$n=2$

$a_2 = 3\cdot 2^{2-1} + 2(-1)^2$ ✓

$= 3\cdot 2 + 2 = 8$

IH

Assume

$a_i = 3\cdot 2^{i-1} + 2(-1)^i$

for $\underbrace{i = 1, \ldots, k.}_{\text{strong induction}}$

IS

$a_{k+1} = a_k + 2a_{k-1}$

$= 3\cdot 2^{k-1} + 2(-1)^k$

by IH $+\, 2\left(3\cdot 2^{k-2} + 2(-1)^{k-1}\right)$

$= 3\cdot 2^{k-1} + 2(-1)^k$

$+\, 3\cdot 2^{k-1} + 4(-1)^{k-1}$?

induction holds

Finding Flaws in Induction Proofs

Induction proofs often tend to be verbose, and it is easy to make flaws in logic that go undetected.

Example

Prove that all people on Earth are the same age. → i.e. in any group of n people, all are same age

Base Case ($n = 1$): Certainly, if we have just one person, they are the same age as everyone else.

this group need not exist!

Induction Hypothesis Assume that in any group of k people, they are all the same age.

e.g. 1, 2, 3, 4, 5

Induction Step
Now, consider a group of $k + 1$ people, $x_1, x_2, \ldots, x_{k+1}$. Let's consider the following two subgroups of k people:

- People $x_1, x_2, \ldots, x_k$ – by the induction hypothesis, they are all the same age
- People $x_2, x_3, \ldots, x_{k+1}$ – again, by the induction hypothesis, they are all the same age

Notice, people $x_2, x_3, \ldots x_k$ are all in both groups. This means, both groups must have the same age, and therefore, we've proven that people $x_1, x_2, \ldots, x_{k+1}$ all have the same age!

Where did we go wrong?

P(1) , P(3) ⇒ P(4), P(4) ⇒ P(5), ...

x_1 x_2

⇓

nobody in both groups

When doing induction, we need to prove the implication

can't assume anything about k

$$P(k) \Rightarrow P(k+1)$$

for **any** arbitrary $k \in \mathbb{N}$. However, our previous proof used an argument that doesn't hold for $P(2)$. Specifically, we used the argument that the "middle individuals" $x_2, x_3, \ldots x_k$ would be in both groups.

However, in the $n = 2$ case, there are two disjoint groups: $\{x_1\}$ and $\{x_2\}$, and there is no overlap between the two groups. The argument that the overlap is in both sets doesn't apply here, because there is no overlap!

$k+1 = i+j \quad 0 \le i,j \le k$

doesn't work when $k=0$

otherwise $i=k, j=1$

Example

Prove that $10n = 0, \forall n \in \mathbb{N}_0$.

Base Case ($n = 0$): $10 \cdot 0 = 0$, therefore the base case holds.

Induction Hypothesis: Assume that $10i = 0$ for all $i \in [0, k]$. → strong induction!

Induction Step

$P(0) \wedge P(1) \wedge P(2) \wedge \ldots \wedge P(k) \Rightarrow P(k+1)$

Now, we must prove $10(k+1) = 0$. We know that we can write $k+1 = i+j$, where i and j are such that $0 \le i, j \le k$. Then,

$$10(k+1) = 10(i+j) = 10i + 10j = 0 + 0 = 0$$

Therefore, induction holds.

doesn't work for $k=0$

$1 = i+j, \quad 0 \le i,j \le 0$

Where did we go wrong?

$P(2) \Rightarrow P(3) \Rightarrow P(4) \Rightarrow \ldots$

↳ just bc implication is true doesn't mean props. individually true not on quiz

section 2.3

Series and Sequences

Now, we'll look at formulas for the sums of arithmetic sequences, as well as sums of the form $\sum_{i=1}^{n} i^k$.

$1+2+3+\ldots+n$

$1^2+2^2+3^2+\ldots+n^2$

$1^3+2^3+\ldots+n^3$

Arithmetic Sequences

initial term

common difference

An arithmetic sequence is defined as

$$t_1 = a \quad t_k = t_{k-1} + d, k \in \mathbb{N}$$

where $d \in \mathbb{R}$. We can also express a general term of an arithmetic sequence without recursion:

$$t_k = a + (k-1)d$$

$t_1 = a$

$t_2 = a + d$

$t_3 = a + 2d$

$t_4 = a + 3d$

$\vdots$

$t_k = a + (k-1)d$

For example,

$$3, \underbrace{10}_{+7}, \underbrace{17}_{+7}, \underbrace{24}_{+7}, \underbrace{31}_{+7}, \underbrace{38}_{+7}, \ldots$$

is an arithmetic sequence with $a = 3$ and $d = 7$.

Now: Suppose we want to determine the sum of the first n terms of an arithmetic sequence, i.e. $\sum_{k=1}^{n}(a + (k-1)d)$.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Sum of First n Natural Numbers

Before determining the sum of an arbitrary arithmetic sequence, let's start with the most basic arithmetic sequence – 1, 2, 3, 4, ... Specifically, we want to find an expression for $\sum_{i=1}^{n} i$.

$$S_n = 1 + 2 + 3 + \dots + n-1 + n$$

$$+ \quad S_n = n + n-1 + n-2 + \dots + 2 + 1$$

$$2S_n = (n+1) + (n+1) + (n+1) + \dots + (n+1) + (n+1)$$

$$2S_n = n(n+1)$$

$$\Rightarrow \boxed{S_n = \frac{n(n+1)}{2}}$$

direct proof,

derivation