Quiz 2

CS 198-087: Introduction to Mathematical Thinking
UC Berkeley EECS
Spring 2019

You will have 30 minutes to work on the quiz. Please fit all of your answers in the space provided. You are not allowed to consult any notes or use any electronics.

Total points: 20

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1. **Pythagorean Triplets** *(Points: 5)*

   Prove that if $a^2 + b^2 = c^2$ for natural numbers a, b, c, then at least one of a, b, c must be even.

Solution:

Let's proceed by contradiction. Let's assume that all of a, b, c are odd. We can then say $a = 2d + 1$, $b = 2e + 1$ and $c = 2f + 1$, where $d, e, f \in \mathbb{Z}$. Then,

$$\begin{aligned} c^2 &= a^2 + b^2 \\ (2f + 1)^2 &= (2d + 1)^2 + (2e + 1)^2 \\ 4f^2 + 4f + 1 &= 4d^2 + 4d + 1 + 4e^2 + 4e + 1 \\ 2(2f^2 + 2f) + 1 &= 2(2d^2 + 2d + 2e^2 + 2e + 1) \end{aligned}$$

On the right hand side, we have an even number (written as $2 \cdot$ (some integer)), while on the left hand side we have an odd number (written as $2 \cdot$ (some integer) $+ 1$). The same number can't be both even and odd, and thus this is a contradiction.

2. **Basic Induction** *(Points: 5)*

   For every natural number n, prove that $3 | (2^{2n} - 1)$. *(Hint: Use induction.)*

Solution: Proof by induction on n.

Base case: For $n = 1$, we have $2^2 - 1 = 3$, so the base case holds.

Inductive hypothesis: For some $n = k$, we have $2^{2k} - 1 = 3m \implies 2^{2k} = 3m + 1$ for some integer m.

Inductive step: For $n = k + 1$, we have:

$$\begin{aligned}
2^{2(k+1)} - 1 &= 2^{2k+2} - 1 \\
&= (2^{2k} \cdot 2^2) - 1 \\
&= (4 \cdot (3m - 1)) - 1 \text{ [from the inductive hypothesis]} \\
&= 12m + 4 - 1 \\
&= 12m + 3 \\
&= 3(4m + 1)
\end{aligned}$$

Thus, the inductive step holds if the induction hypothesis holds.

Since the base case holds and the induction step holds given the inductive hypothesis, the statement is true for all natural numbers $n \geq 1$.

3. **Triangle Inequality** *(Points: 10, 5 each)*

   a. Prove the triangle inequality

$$|a_1 + a_2| \leq |a_1| + |a_2|$$

   for any two real numbers a_1, a_2. *(Hint: Consider four possible cases.)*

   b. Use induction to prove

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n| \quad (1)$$

   holds true for any n. *(Hint: In your induction step, you will need to use the result from part a.)*

Solution:

a. *Note*: By definition of absolute value, we have that when $x < 0$, $|x| = -x$ and when $x \geq 0$, $|x| = x$. This also implies that $x \leq |x|$.

   Now we proceed to construct a proof by cases.

   i. if $(a_1 + a_2) \geq 0$:

$$|a_1 + a_2| = a_1 + a_2 \leq |a_1| + a_2 \leq |a_1| + |a_2|$$

   ii. if $(a_1 + a_2) < 0$:

$$|a_1 + a_2| = -a_1 - a_2 \leq |a_1| + (-a_2) \leq |a_1| + |a_2|$$

b. *Base case*: $|a_1| \leq |a_1|$, for any $a_1 \in \mathbb{R}$, and therefore the base case holds.

   Inductive Hypothesis: Suppose that (1) holds true for some $n = k$.

   Inductive Step: We want to show that (1) true for $n = k+1$. By the triangle inequality, we can see the following:

$$|(a_1 + a_2 + \cdots + a_k) + a_{k+1}| \leq |a_1 + a_2 + \cdots + a_k| + |a_{k+1}|$$

   Based on our inductive hypothesis, we observe that:

$$|a_1 + a_2 + \cdots + a_n + a_{n+1}| \leq |a_1| + |a_2| + \cdots + |a_n| + |a_{n+1}|$$

   Conclusion: Thus we can conclude that (1) holds true for any n.