QUIZ 4

CS 198-087: INTRODUCTION TO MATHEMATICAL THINKING
UC BERKELEY EECS
SPRING 2019

You will have 30 minutes to work on the quiz. Please fit all of your answers in the space provided. You are not allowed to consult any notes or use any electronics.

There are **37** possible points on this quiz, but your score will be out of **27**, and is capped at 100% (e.g. even if you get all 37 points, your score is still 27/27). This allows you to attempt all problems, but doesn't penalize you if you are unable to finish one of them.

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1. *Counting Functions: 9 points (3/3/3)*

   Suppose A, B are two sets.

   a. Suppose $|A| = |B| = n$. How many bijections are there from A to B?

   b. Suppose $|A| = a$ and $|B| = b$. How many functions are there from A to B?

   c. Suppose $|A| = a$ and $|B| = b$, and $a < b$. How many injections are there from A to B?

Solution:

a. Suppose $A = \{x_1, x_2, ..., x_n\}$. There are n options for $f(x_1)$, since x_1 can point to any element in B. Then, there are $n-1$ options for $f(x_2)$, since x_2 can point to any element of B other than $f(x_1)$. There are $n-2$ options for $f(x_3)$, and so on, giving us $n \cdot (n-1) \cdot (n-2) \cdot ... \cdot 2 \cdot 1 = \boxed{n!}$ bijections $A \to B$.

b. Since there are no restrictions on the output, we know that each x_i can map to any element in B. For each element in A, there are b possible outputs. We then have $b \cdot b \cdot ... \cdot b = \boxed{b^a}$ functions $A \to B$.

c. Recall, $f(x)$ is an injection iff $x_1 = x_2 \implies f(x_1) = f(x_2)$. What this means is, no two elements in A can have the same output.

Then, there are b options for $f(x_1)$, $b-1$ options for $f(x_2)$, and so on and so forth, until we have $b-a+1$ options for $f(x_a)$. This gives the product $\boxed{b \cdot (b-1) \cdot (b-2) \cdot \ldots \cdot (b-a+1)}$ which is accepted as an answer.

However, there's a more straightforward way to think of this — our problem is really, how many ways can we select a elements from a group of b, where order matters? That's precisely $\boxed{\dfrac{b!}{(b-a)!}}$. This is equivalent to the other approach.

2. *Permutations: 10 points (3/3/4)*

   Consider the string BIGBILLY.

   a. How many permutations are there of this string?

   b. How many permutations are there of this string, where "BILL" appears as a substring?

   c. How many 3 letter strings are there, consisting solely of the characters in BIGBILLY? *(Hint: Consider two cases: One where all characters are unique, and one where two characters are repeated.)*

Solution:

a. There are 8 characters total, with 2 repeated Bs, 2 repeated Is, and 2 repeated Ls. We start with $8!$ but then need to divide out the repeated characters, thus we're left with $\boxed{\frac{8!}{2!2!2!}}$.

b. Let's abstract "BILL" away to the character "X". We now have have characters, BIGXY, with no repeats. Thus, there are $\boxed{5!}$ permutations of BIGBILLY with BILL as a substring.

c. As mentioned in the hint, there are two cases. Note, there are 5 distinct characters in our string BIGLY.

   Case 1: All characters distinct

   In case 1, our 3 letter string is of the form xyz. There are 5 options for x, 4 for y, and 3 for z, giving a total of $5 \cdot 4 \cdot 3 = 60$ three-letter strings that fall into case 1.

   Case 2: Two repeated characters

   Now, our string is either of the form xxy, xyx or yxx. In each, there are 3 options for x (either B, I, or L) and 4 options for y (anything other than x). There are 3 such cases, therefore the total number of three-letter strings in case 2 is $3 \cdot 4 \cdot 3 = 36$.

   Putting case 1 and 2 together, we have $60 + 36 = \boxed{96}$ three letter strings consisting of characters from BIGBILLY.

3. *Stars and Bars: 12 points (3/3/3/3)*

   Suppose I have 100 \$1 dollar bills that I want to distribute between three of my friends, LeBron, Lonzo and Lance.

   How many ways can this be done...

   a. In general, with no restrictions (other than that everyone receives some non-negative integer amount)?

   b. If everyone receives at least \$1?

   c. If everyone receives at least \$t, for $0 \leq t \leq 33$?

   d. Such that LeBron and Lonzo receive the same amount? (*Hint: How can we format this as solving the number of solutions to $x + y = 50$?*)

Solution: We will model each question as finding the number of non-negative integer solutions to $x_1 + x_2 + x_3 = 100$, with different sets of constraints in each. Let x_1 represent LeBron, x_2 Lonzo and x_3 Lance.

a. Here, our only constraints are $x_1, x_2, x_3 \geq 0$. This is given by the standard stars-and-bars solution of $\binom{100+2}{2} = \boxed{\binom{102}{2}}$, since we have 100 stars and 2 bars.

b. Defining $x_i' = x_i - 1$ gives us $x_1' + x_2' + x_3' = 97$, which has $\binom{97+2}{2} = \boxed{\binom{99}{2}}$ solutions.

c. Now, we define $x_i' = x_i - t$, for $0 \leq x \leq 33$. Then:

$$\begin{aligned} x_1 + x_2 + x_3 &= 100 \\ (x_1 - t) + (x_2 - t) + (x_3 - t) &= 100 - 3t \\ x_1' + x_2' + x_3' &= 100 - 3t \end{aligned}$$

   which has $\boxed{\binom{102-3t}{2}}$ solutions.

d. Now, we set $x_1 = x_2$, meaning we are looking at $2x_1 + x_3 = 100$. Since $2x_1$ is an even number, and 100 is even, we know that x_3 must also be even. So, we set $x_3 = 2k$, for some integer k, where $0 \leq k \leq 50$. We are now looking at the number of non-negative integer solutions to $x_1 + k = 50$, which can be modelled using 50 stars and 1 bar. This has $\binom{50+1}{1} = \boxed{51}$ solutions.

4. *Combinatorial Proof: 6 points*

Give a **combinatorial proof** of the following identity (assuming $n \geq r \geq k$):

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

(Note: Remember, combinatorial proofs are arguments that both the LHS and RHS count the same quantity. Algebraic proofs, i.e. ones that manipulate one side of this equation and show that it is equal to the other, will be given no credit.)

Solution:

Suppose I have n basketball players, from which I want to select a team of r. Furthermore, k of my r players will be considered "captains."

LHS: I could first select all of my r players, which I can do in $\binom{n}{r}$ ways, and then select my k captains from those r, which I can do in $\binom{r}{k}$ ways, giving a total of $\binom{n}{r}\binom{r}{k}$.

RHS: I could also first select my k captains, which I can do in $\binom{n}{k}$ ways, and then select my remaining $r-k$ players from the general pool, which I can do in $\binom{n-k}{r-k}$ ways, giving a total of $\binom{n}{k}\binom{n-k}{r-k}$ ways.

Therefore, $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$.