QUIZ 5

CS 198-087: INTRODUCTION TO MATHEMATICAL THINKING
UC BERKELEY EECS
SPRING 2019

You will have 30 minutes to work on the quiz. Please fit all of your answers in the space provided. You are not allowed to consult any notes or use any electronics.

There are **20** possible points on this quiz.

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1. *Binomial Theorem Fundamentals (Points: 2 + 3 + 3 = 8)*

Suppose $p_{a,n}(x) = (ax^2 - (a+1))^n$.

a. Determine the sum of the coefficients of $p_{2,n}(x)p_{5,n}(x)$ for odd n.

b. Determine the general term in the expansion of $p_{5,9}(x)$.

c. Determine the coefficient of x^{10} in $p_{5,9}(x)$.

Solution:

a. Recall, the sum of the coefficients of a polynomial expansion is attained when setting all variables to 1. Then,

$$\begin{aligned} p_{2,n}(1)p_{5,n}(1) &= (2 \cdot 1^2 - 3)^n (5 \cdot 1^2 - 6)n \\ &= (-1)^n(-1)^n \\ &= (-1)^{2n} \\ &= \boxed{1} \end{aligned}$$

Note: It didn't matter that we specified n was odd, since $(-1)^{2n} = 1$ for any integer n.

b.

$$p_{5,9}(x) = (5x^2 - 6)^9$$

Then,

$$\begin{aligned} t_k &= \binom{9}{k}(5x^2)^{9-k}(-6)^k \\ &= \boxed{(-1)^k \binom{9}{k} 5^{9-k} 6^k x^{18-2k}} \end{aligned}$$

c. Now, we must set the exponent on x in the general term equal to 10, and determine the corresponding term.

$$18 - 2k = 10 \implies k = 4$$

Then,

$$\begin{aligned} t_4 &= (-1)^4 \binom{9}{4} 5^{9-4} 6^4 x^{10} \\ &\implies \boxed{\text{coef}(x^{10}) = \binom{9}{4} 5^5 6^4} \end{aligned}$$

2. *Binomial Approximations (Points: 5)*

Use the first three terms of the binomial expansion to approximate $\sqrt[3]{124}$.

Note: Your answer should be of the form

$$a_0 \cdot 5 + \frac{a_1}{5} + \frac{a_2}{5^2} + \frac{a_3}{5^3} + \frac{a_4}{5^4} + \frac{a_5}{5^5}$$

Your job is to determine $a_0, a_1, a_2, a_3, a_4, a_5$. In other words, don't try and simplify your answer to a decimal.

Following the technique used in class, your answer will naturally be in this format. Note, one or more of a_i may be equal to 0.

Solution:

Notice $124 = 125 - 1$, and so we are looking at the first three terms of $(125-1)^{\frac{1}{3}}$.

$$\begin{aligned}(125-1)^n &= \binom{n}{0}125^n - \binom{n}{1}125^{n-1} + \binom{n}{2}125^{n-2} + ... \\ &= 125^n - n125^{n-1} + \frac{n(n-1)}{2}125^{n-2} + ... \\ (125-1)^{\frac{1}{3}} &\approx 125^{\frac{1}{3}} - \frac{1}{3}125^{-\frac{2}{3}} + \frac{(1/3)(-2/3)}{2}125^{-\frac{5}{3}} \\ &= 5 - \frac{1}{3} \cdot \frac{1}{5^2} - \frac{1}{9} \cdot \frac{1}{5^5}\end{aligned}$$

Then, we have

$$\boxed{a_0 = 1, a_2 = -\frac{1}{3}, a_5 = -\frac{1}{9}, a_1 = a_3 = a_4 = 0}$$

For reference, our approximation yields 4.986631111...,

whereas the true value is 4.9866309522...; our approximation is accurate for the first 5 digits. Not bad!

3. *Root Relations (Points: 3 + 4 = 7)*

Suppose $f(x) = x^2 - \alpha x + \beta$, and suppose $f(x)$ has roots r_1, r_2.

a. Determine $r_1^2 r_2 + r_1 r_2^2$.

b. Find a polynomial with roots r_1^2, r_2^2.

Solution:

a. By Vieta's, $r_1 + r_2 = \alpha$ and $r_1 r_2 = \beta$.

Then, notice we can factor $r_1^2 r_2 + r_1 r_2^2$ as $r_1 r_2 (r_1 + r_2)$, and thus our result is $\boxed{\alpha\beta}$.

b.

$$\begin{aligned}(r_1 + r_2)^2 &= r_1^2 + r_2^2 + 2r_1 r_2 \\ \alpha^2 &= r_1^2 + r_2^2 + 2\beta \\ \implies r_1^2 + r_2^2 &= \alpha^2 - 2\beta\end{aligned}$$

$$\begin{aligned}r_1^2 r_2^2 &= (r_1 r_2)^2 \\ \implies r_1^2 r_2^2 &= \beta^2\end{aligned}$$

Therefore, the polynomial $\boxed{x^2 - (\alpha^2 - 2\beta)x + \beta^2}$ has roots r_1^2, r_2^2, as required.